Resonances for non-analytic potentials

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May 12, 2008

Abstract

We consider semiclassical Schrödinger operators on $\mathbb{R}^n$, with C^∞ potentials decaying polynomially at infinity. The usual theories of resonances do not apply in such a non-analytic framework. Here, under some additional conditions, we show that resonances are invariantly defined up to any power of their imaginary part. The theory is based on resolvent estimates for families of approximating distorted operators with potentials that are holomorphic in narrow complex sectors around $\mathbb{R}^n$.

1 Introduction

In physics, the notion of quantum resonance has appeared at the begining of quantum mechanics. Its introduction was motivated by the behavior of various quantities related to scattering experiments, such as the scattering cross-section. At certain energies, these quantities present peaks (nowaday called Breit-Wigner peaks), which were modelized by a Lorentzian shaped function

$$w_{a,b} : \lambda \mapsto ((\lambda - a)^2 + b^2)^{-1}.$$

The real numbers a and b stand for the location of the maximum of the peak and its height. Of course for $\rho = a - ib \in \mathbb{C}$, one has

$$w_{a,b}(\lambda) = \frac{1}{|\lambda - \rho|^2},$$

and the complex number ρ was called a resonance. Such complex values for energies had also appeared for example in the work [7] by Gamow, to explain α-radioactivity, and were associated to the existence of some decaying state of energy $a = \mathrm{Re}\,\rho$ and lifetime $1/b = 1/|\mathrm{Im}\,\rho|$.

However, these complex numbers are not defined in a completely exact way, in the sense that the peaks do not perceivably change if these numbers are modified by a quantity much

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smaller than their imaginary part. Indeed, a straightforward computation shows that the relative difference between such two peaks $w_{a,b}$ and $w_{a',b'}$ verifies,

$$\sup_{\lambda\in\mathbb{R}}\left|\frac{w_{a,b}(\lambda)-w_{a',b'}(\lambda)}{w_{a',b'}(\lambda)}\right|\leq 2\left|\frac{\rho-\rho'}{\mathrm{Im}\,\rho}\right|+\left|\frac{\rho-\rho'}{\mathrm{Im}\,\rho}\right|^2$$

where we have also set $\rho'=a'-ib'$. As a consequence, the two peaks become undistinguishable if $|\rho-\rho'|<<|\mathrm{Im}\,\rho|$, that is, there is no physical relevance to associate the resonance $\rho=a-ib$ to $w_{a,b}$ rather than any other ρ' verifying $|\rho-\rho'|<<|\mathrm{Im}\,\rho|$. Notice also that the more the resonance is far from the real line, the more irrelevant this precision becomes.

On the mathematical side, the more recent theory of resonances for Schrödinger operators has permitted to give a rigorous framework and to obtain very precise results, in particular on the location of resonances in relation with the geometry of the underlying classical flow. However, it is based on the notion of complex scaling, in more and more sophisticated versions (see, e.g., [1, 2, 20, 11, 19, 4, 16, 17, 10]) that all require analyticity assumptions on the potential (or its Fourier transform).

There is a small number of works about the definition of resonances for non-analytic potentials, as e.g. [18, 9, 21, 12, 3]. In [18, 9, 21, 12], the point of view is quite different from ours, while in [3], the definition is based on the use of an almost-analytic extension of the potential and seems to strongly depend both on the choice of this extension and on the complex distortion.

Here our purpose is to give a definition that fulfills both the mathematical requirement of being invariant with respect to the choices one has to make, and the physical requirement of being more accurate as the resonance become closer to the real (or, equivalently, as the Breit-Wigner peak becomes narrower). Dropping the physically irrelevant precision for the definition of resonances, we can also drop the spurious assumption on the analyticity of the potential.

More precisely, we associate to a Schrödinger operator P a discrete set $\Lambda\subset\mathbb{C}$ with certain properties, such that, for any other set Λ' with the same properties, there exists a bijection $B:\Lambda'\to\Lambda$ with $B(\rho)-\rho=\mathcal{O}(|\mathrm{Im}\,\rho|^\infty)$ uniformly. The set of resonances of P is the corresponding equivalence class of Λ. Of course, when the potential is dilation analytic at infinity, we recover the usual set of resonances up to the same error $\mathcal{O}(|\mathrm{Im}\,\rho|^\infty)$.

The properties characterizing Λ basically involve the resonances of a (essentially arbitrary) family of dilation-analytic operators $(P^\mu)_{0<\mu\leq\mu_0)}$, such that,

$$\begin{aligned}&P^\mu \text{ is dilation-analytic in a complex sector of angle } \mu \text{ around } \mathbb{R}^n;\\ &\|P^\mu-P\|=\mathcal{O}(\mu^\infty) \text{ uniformly as } \mu\to 0_+,\end{aligned}$$

and the constructive proof of the existence of the set Λ mainly consists in studying such a family and, in particular, in obtaining resolvent estimates uniform in μ.

In this paper, we address the case of an isolated cluster of resonances with a bounded (with respect to h) cardinality. We hope to treat the general case elsewhere, as well as to give a detailed description of the quantum evolution $e^{itP/h}=e^{itP^\mu/h}+\mathcal{O}(|t|h^{-1}\mu^\infty)$ in terms of the resonances in Λ.

The paper is organized as follows. We give our assumptions and state our main results in Section 2. Then, in Section 3, we give two paradigmatic situations where our constructions

apply: the non-traping case and the shape resonances case. In section 4 we present a suitable notion of analytic approximation of a C^∞ function through which we define the operator P^μ. In Section 5 we show that a properly defined analytic distorted operator P^μ_θ of the latter verifies a nice resolvent estimate in the upper half complex plane even very near to the real axis. The sections 6,7 and 8 are devoted to the proof of Theorem 2.1, Theorem 2.2 and Theorem 2.5 respectively. We construct the set of resonances Λ, and prove Theorem 2.6 in Section 9. In the last Section 10, we prove our statements concerning the shape resonances. Eventually, we have placed in Appendix A the proofs of two technical lemmas.

2 Notations and Main Results

We consider the semiclassical Schrödinger operator,

$$P = -h^2\Delta + V,$$

where $V = V(x)$ is a real smooth function of $x \in \mathbb{R}^n$, such that,

$$\partial^\alpha V(x) = \mathcal{O}(\langle x\rangle^{-\nu-|\alpha|}), \tag{2.1}$$

for some $\nu > 0$ and for all $\alpha \in \mathbb{Z}^n_+$. We also fix $\widetilde{\nu} \in (0,\nu)$ once for all, and, for any $\mu > 0$ small enough, we denote by V^μ a $|x|$-analytic $(\mu,\widetilde{\nu})$-approximation of V in the sense of Section 4. In particular, V^μ is analytic with respect to $r = |x|$ in $\{r \geq 1\}$, it can be extended into a holomorphic function of r in the sector $\Sigma := \{\mathrm{Re}\, r \geq 1\,,\, |\mathrm{Im}\, r| \leq 2\mu \mathrm{Re}\, r\}$, and it verifies,

$$V^\mu(x) - V(x) = \mathcal{O}(\mu^\infty \langle x\rangle^{-\widetilde{\nu}}), \tag{2.2}$$

uniformly on $\mathbb{R}^n$. (See Section 4 for more properties of V^μ.)

Then, for any $\theta \in (0,\mu]$, the operator,

$$P^\mu := -h^2\Delta + V^\mu, \tag{2.3}$$

can be distorded analytically into,

$$P^\mu_\theta := U_\theta P^\mu U_\theta^{-1}, \tag{2.4}$$

where U_θ is any transformation of the type,

$$U_\theta\varphi(x) := \varphi(x + i\theta A(x)), \tag{2.5}$$

with $A(x) := a(|x|)x$, $a \in C^\infty(\mathbb{R}_+)$, $a = 0$ near 0, $0 \leq a \leq 1$ everywhere, $a(|x|) = 1$ for $|x|$ large enough. The essential spectrum of P^μ_θ is $e^{-2i\theta}\mathbb{R}$, and its discrete spectrum $\sigma_{disc}(P^\mu_\theta)$ is included in the lower half-plane and does not depend on the choice of the function a. Moreover, it does not depend on θ, in the sense that for any $\theta_0 \in (0,\mu]$, and any $\theta \in [\theta_0,\mu]$, one has,

$$\sigma_{disc}(P^\mu_\theta) \cap \Sigma_{\theta_0} = \sigma_{disc}(P^\mu_{\theta_0}) \cap \Sigma_{\theta_0},$$

where we have set $\Sigma_{\theta_0} := \{z \in \mathbb{C}\,;\, -2\theta_0 < \arg z \leq 0\}$ (observe that one also has $\sigma_{disc}(P^\mu_\theta) = \sigma_{disc}(\widetilde{U}_\theta P^\mu \widetilde{U}_\theta^{-1})$, where $\widetilde{U}_\theta\varphi(x) := \sqrt{\det(Id + i\theta\, {}^t dA(x))}\varphi(x+i\theta A(x))$ is an analytic distorsion more widely used in the literature).

We denote by,

$$\Gamma(P^\mu) := \sigma_{disc}(P^\mu_\mu) \cap \Sigma_\mu,$$

the set of resonances of P^μ counted with their multiplicity. In what follows, we also use the following notation: If E and E' are two h-dependent subsets of $\mathbb{C}$, and $\alpha = \alpha(h)$ is a h-dependent positive quantity that tends to 0 as h tends to 0_+, we write,

$$E' = E + \mathcal{O}(\alpha),$$

when there exists a constant $C > 0$ (uniform with respect to all other parameters) and a bijection

$$b : E' \to E,$$

such that,

$$|b(\lambda) - \lambda| \leq C\alpha$$

for all $h > 0$ small enough.

Now, we fix some energy level $\lambda_0 > 0$, and a constant $\delta > 0$. For any h-dependent numbers $\widetilde{\mu}(h), \mu(h)$, and any h-dependent bounded intervals $I(h), J(h)$, verifying,

$$0 < \widetilde{\mu}(h) \leq \mu(h) \leq h^\delta; \tag{2.6}$$

$$I(h) \subset J(h)\ ;\ \ \mathrm{diam}(J \cup \{\lambda_0\}) \leq h^\delta, \tag{2.7}$$

we consider the following property:

$$\mathcal{P}(\widetilde{\mu}, \mu; I, J) : \qquad \begin{cases} \mathrm{Re}\,(\Gamma(P^\mu) \cap (J - i[0, \lambda_0\widetilde{\mu}])) \subset I; \\ \#(\Gamma(P^\mu) \cap (J - i[0, \lambda_0\widetilde{\mu}])) \leq \delta^{-1}; \\ \mathrm{dist}\,(I, \mathbb{R} \setminus J) \geq h^{-\delta}\omega_h(\widetilde{\mu}), \end{cases}$$

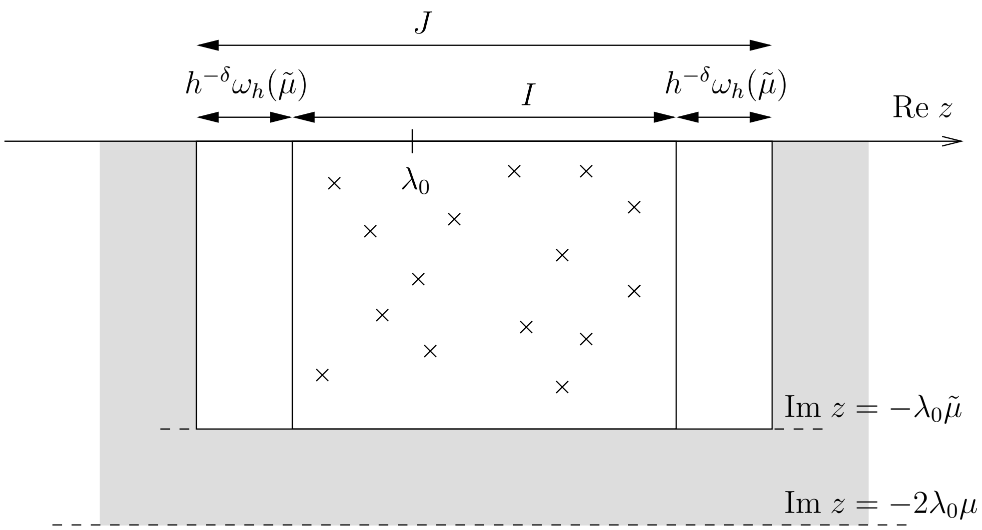

Figure 1: The property $\mathcal{P}(\widetilde{\mu}, \mu; I, J)$.

where, for $\theta > 0$, we have set,

$$\omega_h(\theta) := \theta \left(\ln \frac{1}{\theta} + h^{-n} (\ln \frac{1}{h})^{n+1} \right)^{1/2}.$$

Notice that by (2.7), the property $\mathcal{P}(\widetilde{\mu}, \mu; I, J)$ implies $\omega_h(\widetilde{\mu}) \leq h^{2\delta}$.

Theorem 2.1. *Suppose $\mathcal{P}(\widetilde{\mu}, \mu; I, J)$ holds for some $\widetilde{\mu}, \mu, I$ and J verifying (2.6) – (2.7). Then for all $\theta \in]0, \widetilde{\mu}]$, there exists an interval*

$$J' = J + \mathcal{O}(\omega_h(\theta)),$$

such that,

$$\|(P_\theta^\mu - z)^{-1}\| \leq C\theta^{-C} \prod_{\rho \in \Gamma(\widetilde{\mu}, \mu, J)} |z - \rho|^{-1},$$

for all $z \in J' + i[-C\theta h^{n_1}, C\theta h^{n_1}]$. Here we have set $n_1 := n + \delta$,

$$\Gamma(\widetilde{\mu}, \mu, J) := \Gamma(P^\mu) \cap (J - i[0, \lambda_0 \widetilde{\mu}]),$$

and $C > 0$ is a constant independent of $\widetilde{\mu}$, μ, θ, I and J.

Thanks to this result, one can compare the resonances of the operators P^μ for different values of μ, as follows:

Theorem 2.2. *Let $N_0 \geq 1$ be a constant. Suppose $\mathcal{P}(\widetilde{\mu}, \mu; I, J)$ holds for some $\widetilde{\mu}, \mu, I$ and J verifying (2.6) – (2.7), and that $\widetilde{\mu} > \mu^{N_0}$. Then, for any $\theta \in [\mu^{N_0}, \widetilde{\mu}]$, there exist an interval,*

$$J' = J + \mathcal{O}(\omega_h(\theta))$$

and $\tau \in [h^{n_1}\theta, 2h^{n_1}\theta]$, such that, for any constant $N_1 \geq 1$ and any $\mu' \in [\mu^{N_1}, \mu^{1/N_1}]$ with $\theta \leq \mu'$, one has,

$$\Gamma(P^{\mu'}) \cap (J' - i[0, \tau]) = \Gamma(P^\mu) \cap (J' - i[0, \tau]) + \mathcal{O}(\mu^\infty).$$

Remark 2.3. *The only properties of V^μ used in the proof of this result are that V^μ is a holomorphic function of r in the sector $\Sigma := \{\mathrm{Re}\, r \geq 1\,, |\mathrm{Im}\, r| \leq 2\mu \mathrm{Re}\, r\}$, and it verifies (2.2) and (4.2) for some $\tilde{\nu} > 0$. In particular, the proof also shows that, up to $\mathcal{O}(\mu^\infty)$, the set $\Gamma(P^\mu)$ does not depend on any particular choice of V^μ.*

Remark 2.4. *As we will see in the proof, the condition $\tau \in [h^{n_1}\theta, 2h^{n_1}\theta]$ can actually be replaced by $\tau \in [h^{n_1}\theta, h^{n_1}\theta + (h^{n_1}\theta)^M]$, for any fixed $M \geq 1$.*

We also show that the validity of $\mathcal{P}(\widetilde{\mu}, \mu; I, J)$ persists when decreasing $\widetilde{\mu}$ and μ suitably, up to a small change of I and J.

Theorem 2.5. *Suppose $\mathcal{P}(\widetilde{\mu}, \mu; I, J)$ holds for some $\widetilde{\mu}, \mu, I$ and J verifying (2.6) – (2.7). Assume furthermore that there is a constant $N_0 \geq 1$ with $\widetilde{\mu} \geq \mu^{N_0}$. Then, there exist two intervals,*

$$I' = I + \mathcal{O}(\mu^\infty);$$
$$J' = J + \mathcal{O}(\omega_h(\widetilde{\mu})),$$

such that $\mathcal{P}(h^{n_1}\mu', \mu'; I', J')$ holds, for any $\mu' \in (0, \widetilde{\mu}]$.

Finally, the following result gives a definition of resonances for P, up to any power of their imaginary part.

Theorem 2.6. *Suppose $\mathcal{P}(\widetilde{\mu}, \mu; I, J)$ holds for some $\widetilde{\mu}, \mu, I$ and J verifying (2.6) – (2.7). Assume furthermore that there is a constant $N_0 \geq 1$ with $\widetilde{\mu} \geq \mu^{N_0}$. Then, there exist,*

$$\begin{aligned} &\text{an interval} && I' = I + \mathcal{O}(\mu^\infty); \\ &\text{an interval} && J' = J + \mathcal{O}(\omega_h(\widetilde{\mu})); \\ &\text{a discrete set} && \Lambda \subset I' - i[0, 2h^{2n_1}\widetilde{\mu}], \end{aligned}$$

such that,

$$(\star) \quad \left| \begin{array}{l} \text{for any } \mu' \in (0, \widetilde{\mu}], \text{ there exist } \tau \in [h^{2n_1}\mu', 2h^{2n_1}\mu'] \text{ with,} \\ \qquad \Gamma(P^{\mu'}) \cap (J' - i[0,\tau]) = \Lambda \cap (J' - i[0,\tau]) + \mathcal{O}((\mu')^\infty). \end{array} \right.$$

Moreover, any other set $\widetilde{\Lambda} \subset I' - i[0, 2h^{2n_1}\widetilde{\mu}]$ verifying ($\star$), possibly with some other choice of V^μ, is such that there exist $\tau' \in [\frac{1}{2}h^{2n_1}\widetilde{\mu}, h^{2n_1}\widetilde{\mu}]$ and a bijection,

$$B : \Lambda \cap (J' - i[0,\tau']) \to \widetilde{\Lambda} \cap (J' - i[0,\tau']),$$

with,

$$B(\lambda) - \lambda = \mathcal{O}(|\mathrm{Im}\,\lambda|^\infty).$$

The set Λ will be called the set of resonances of P in $J' - i[0, \frac{1}{2}h^{2n_1}\widetilde{\mu}]$. Here we adopt the convention that real elements of Λ are counted with a positive integer multiplicity in the natural way (see Section 9).

3 Two examples

Here, we describe two explicit situations where the previous results apply.

3.1 The non-trapping case

We suppose first that the energy λ_0 is non-trapping, i.e. for any $(x,\xi) \in p^{-1}(\lambda_0)$ we have

$$|\exp tH_p(x,\xi)| \to \infty \text{ as } |t| \to \infty,$$

where $p(x,\xi) := \xi^2 + V(x)$ is the principal symbol of P, and $H_p = \partial_\xi p \partial_x - \partial_x p \partial_\xi$ is the Hamilton field of p.

Then the result of [13] can be applied to P^μ with $\mu = Ch\ln(h^{-1})$ for any arbitrary constant $C > 0$, and tells us that P^μ has no resonances in $[\lambda_0 - 2\varepsilon, \lambda_0 + 2\varepsilon] - i[0, \lambda_0\mu]$ with some $\varepsilon > 0$ constant. In that case, for any $\delta > 0$, $\mathcal{P}(h^{n_1}\mu, \mu; I, J)$ is verified with $I = [\lambda_0 - h^\delta, \lambda_0 + h^\delta]$ and $J = [\lambda_0 - 2h^\delta, \lambda_0 + 2h^\delta]$, and the previous results tell us that P has no resonances in $I - i[0, \frac{1}{2}h^{3n_1}\mu]$ in the sense of Theorem 2.6.

3.2 The shape resonances

Now we assume instead that, in addition to (2.1), the potential V presents the geometric configuration of the so-called "point-well in an island", as described in [10]. More precisely, we suppose

$$
\text{(H)} \qquad \begin{cases} \text{There exist a connected bounded open set } \ddot{\mathrm{O}} \subset \mathbb{R}^n, \text{ and } x_0 \in \ddot{\mathrm{O}}, \text{ such that,} \\ \bullet\ \lambda_0 := V(x_0) > 0\ ;\ V > \lambda_0 \text{ on } \ddot{\mathrm{O}}\backslash\{x_0\}\ ;\ \nabla^2 V(x_0) > 0\ ;\ V = \lambda_0 \text{ on } \partial\ddot{\mathrm{O}}; \\ \bullet\ \text{Any point of } \{(x,\xi) \in \mathbb{R}^{2n}\ ;\ x \in \mathbb{R}^n\backslash\ddot{\mathrm{O}}\ ,\ \xi^2 + V(x) = \lambda_0\} \text{ is non-trapping.} \end{cases}
$$

We denote by $(e_k)_{k\geq 1}$ the increasing sequence of (possibly multiple) eigenvalues of the harmonic oscillator $H_0 = -\Delta + \frac{1}{2}\langle V''(x_0)x, x\rangle$. We have

Theorem 3.1. *Assume (2.1) and (H). Then, for any $k_0 \geq 1$ and any $\delta > 0$, $\mathcal{P}(\widetilde{\mu}, \mu; I, J)$ holds with*

$$\mu = h^\delta \quad ; \quad \widetilde{\mu} = h^{\max(\frac{n}{2},1)+1+\delta},$$

and

$$I = [\lambda_0 + (e_1 - \varepsilon)h, \lambda_0 + (e_{k_0} + \varepsilon)h] \quad ; \quad J = [\lambda_0, \lambda_0 + (e_{k_0+1} - \varepsilon)h],$$

where $\varepsilon > 0$ is any fixed number in $(0, \min(\frac{e_1}{2}, \frac{e_{k_0+1}-e_{k_0}}{3})]$.

Actually, we prove in Section 10 that any resonance ρ of P^μ in $J - i[0, \widetilde{\mu}]$ is such that there exists $k \leq k_0$ with

$$\mathrm{Re}\, \rho - (\lambda_0 + e_k h) = \mathcal{O}(h^{3/2}),$$

and

$$\mathrm{Im}\, \rho = \mathcal{O}(e^{-2S_1/h}),$$

where $S_1 > 0$ is any number less than the Agmon distance between x_0 and $\partial\ddot{\mathrm{O}}$. Recall that the Agmon distance is the pseudo-distance associated to the degenerate metric $(V(x) - \lambda_0)_+ dx^2$.

More generally, if $\mu' \in [e^{-\eta/h}, \mu]$ with $\eta > 0$ small enough, we prove that any resonance ρ of $P^{\mu'}$ in $J - i[0, \lambda_0 \min(\mu', h^{2+\delta})]$, verifies

$$\mathrm{Re}\, \rho - (\lambda_0 + e_k h) = \mathcal{O}(h^{3/2}),$$

for some $k \leq k_0$, and

$$\mathrm{Im}\, \rho = \mathcal{O}(e^{-2(S_0-\eta)/h}).$$

Applying Theorem 2.6 with $\mu' = e^{-\eta/h}$ $(0 < \eta < S_0)$, we deduce that the resonances of P in $[\lambda_0, \lambda_0 + Ch] - i[0, \frac{1}{2}h^{2n+\max(\frac{n}{2},1)+1+3\delta}]$ satisfy the same estimates.

4 Preliminaries

In this section, following an idea of [6], we define and study the notion of analytic $(\mu, \widetilde{\nu})$-approximations.

Definition 4.1. *For any $\mu > 0$ and $\widetilde{\nu} \in (0, \nu)$, we say that a real smooth function V^μ on $\mathbb{R}^n$ is a $|x|$-analytic $(\mu, \widetilde{\nu})$-approximation of V, if V^μ is analytic with respect to $r = |x|$ in $\{r \geq 1\}$, V^μ can be extended into a holomorphic function of r in the sector $\Sigma(2\mu) := \{\mathrm{Re}\, r \geq 1\,,\ |\mathrm{Im}\, r| < 2\mu \mathrm{Re}\, r\}$, and, for any multi-index α, it verifies,*

$$\partial^\alpha (V^\mu(x) - V(x)) = \mathcal{O}(\mu^\infty \langle x\rangle^{-\widetilde{\nu}-|\alpha|}), \tag{4.1}$$

uniformly with respect to $x \in \mathbb{R}^n$ and $\mu > 0$ small enough, and,

$$\partial^\alpha V^\mu(x) = \mathcal{O}(\langle \mathrm{Re}\, x\rangle^{-\widetilde{\nu}-|\alpha|}), \tag{4.2}$$

uniformly with respect to $x \in \Sigma(2\mu)$ and $\mu > 0$ small enough.

Then, we have,

Proposition 4.2. *Let $V = V(x)$ be a real smooth function of $x \in \mathbb{R}^n$ verifying (2.1). Then, one has,*

(i) For any $\mu > 0$ and $\widetilde{\nu} \in (0, \nu)$, there exists a $|x|$-analytic $(\mu, \widetilde{\nu})$-approximation of V;

(ii) If V^μ and W^μ are two $|x|$-analytic $(\mu, \widetilde{\nu})$-approximations of V, then, for all $\alpha \in \mathbb{N}^n$, one has,

$$\partial^\alpha (V^\mu(x) - W^\mu(x)) = \mathcal{O}(\mu^\infty \langle \mathrm{Re}\, x\rangle^{-\widetilde{\nu}-|\alpha|}),$$

uniformly with respect to $x \in \Sigma(\mu)$ and $\mu > 0$ small enough.

Proof. We denote by $\widetilde{V}$ a smooth function on $\mathbb{C}^n$ verifying,

- $\widetilde{V} = V$ on $\mathbb{R}^n$;
- For any $C > 0$, one has,

$$\overline{\partial}\widetilde{V} = \mathcal{O}\left((|\mathrm{Im}\, x|/\langle \mathrm{Re}\, x\rangle)^\infty \langle \mathrm{Re}\, x\rangle^{-\nu}\right),$$

  uniformly on $\{|\mathrm{Im}\, x| \leq C\langle \mathrm{Re}\, x\rangle\}$;
- For any $C > 0$ and $\alpha \in \mathbb{N}^n$, one has,

$$\partial^\alpha \widetilde{V} = \mathcal{O}\left(\langle \mathrm{Re}\, x\rangle^{-\nu-|\alpha|}\right),$$

  uniformly on $\{|\mathrm{Im}\, x| \leq C\langle \mathrm{Re}\, x\rangle\}$.

Note that such a function $\widetilde{V}$ (called an "almost-analytic" extension of V: See, e.g., [15]) can easily be obtained by taking a resummation of the formal series,

$$\widetilde{V}(x) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{i^{|\alpha|}(\mathrm{Im}\, x)^\alpha}{\alpha!}\partial^\alpha V(\mathrm{Re}\, x). \tag{4.3}$$

Indeed, since we have $\partial^\alpha V(\mathrm{Re}\, x) = \mathcal{O}(\langle \mathrm{Re}\, x\rangle^{-\nu-|\alpha|})$, the resummation is well defined up to $\mathcal{O}\left((|\mathrm{Im}\, x|/\langle \mathrm{Re}\, x\rangle)^\infty \langle \mathrm{Re}\, x\rangle^{-\nu}\right)$, and the standard procedure of resummation (see, e.g., [5, 14]) also gives the required estimates on the derivatives of $\widetilde{V}$. Conversely, by a Taylor expansion, we see that any $\widetilde{V}$ verifying the required conditions is necessarily a resummation of the series (4.3).

Now, if V^μ is a $|x|$-analytic $(\mu,\widetilde{\nu})$-approximation of V, then, for any $x = r\omega \in \Sigma(\mu)$ ($\omega \in S^{n-1}$) and $N \geq 0$, we have,

$$\begin{aligned}
V^\mu(x) - \widetilde{V}(x) &= \sum_{k=0}^{N} \frac{i^k(\mathrm{Im}\, r)^k}{k!}\partial_r^k V^\mu(\mathrm{Re}\, r\cdot\omega) \\
&\quad + \frac{(i\mathrm{Im}\, r)^{N+1}}{(N+1)!}\int_0^1 \partial_r^{N+1}(V^\mu((\mathrm{Re}\, r + it\mathrm{Im}\, r)\cdot\omega))dt - \widetilde{V}(x) \\
&= \sum_{k=0}^{N} \frac{i^k(\mathrm{Im}\, r)^k}{k!}\partial_r^k (V^\mu(\mathrm{Re}\, x) - V(\mathrm{Re}\, x)) + \mathcal{O}(\mu^{N+1}\langle \mathrm{Re}\, x\rangle^{-\widetilde{\nu}}) \\
&= \mathcal{O}(\mu^\infty\langle \mathrm{Re}\, x\rangle^{-\widetilde{\nu}}) + \mathcal{O}(\mu^{N+1}\langle \mathrm{Re}\, x\rangle^{-\widetilde{\nu}}),
\end{aligned}$$

and, similarly, for any $\alpha \in \mathbb{N}^n$,

$$\partial^\alpha(V^\mu(x) - \widetilde{V}(x)) = \mathcal{O}(\mu^\infty\langle \mathrm{Re}\, x\rangle^{-\widetilde{\nu}-|\alpha|}).$$

In particular, we have proved (ii).

Now, we proceed with the construction of such a V^μ.

For $x \in \mathbb{R}^n\backslash 0$, we set $\omega = \frac{x}{|x|}$, $r = |x|$, and $s = \ln r$. In particular, for any t real small enough, the dilation $x \mapsto e^t x$ becomes $(s,\omega) \mapsto (s+t,\omega)$ in the new coordinates (s,ω).

For $\omega \in S^{n-1}$ and $s \in \mathbb{C}$ with $|\mathrm{Im}\, s|$ small enough, we set $\widetilde{V}_1(s,\omega) := \widetilde{V}(e^s\omega)$, where $\widetilde{V}$ is an almost-analytic extention of V as before. Then, for $|\mathrm{Im}\, s| < 2\mu$ and $\mathrm{Re}\, s \geq -\mu$, we define,

$$V_1^\mu(s,\omega) := \frac{e^{-\widetilde{\nu}s}}{2i\pi}\int_\gamma \frac{e^{\widetilde{\nu}s'}\widetilde{V}_1(s',\omega)}{s-s'}ds', \tag{4.4}$$

where γ is the oriented complex contour,

$$\gamma := ((+\infty, -2\mu] + 2i\mu) \cup (-2\mu + 2i[\mu, -\mu]) \cup ([-2\mu, +\infty) - 2i\mu). \tag{4.5}$$

Observe that, by construction, we have $\widetilde{V}_1(s',\omega) = \mathcal{O}(e^{-\nu\mathrm{Re}\, s'})$, so that the previous integral is indeed absolutely convergent. Therefore, the (s,ω)-smoothness and s-holomorphy of V_1^μ are obvious consequences of Lebesgue's dominated convergence theorem. Since γ is symmetric with respect to $\mathbb{R}$, we also have that $V_1^\mu(s,\omega)$ is real for s real. Moreover, since $|s-s'| \geq \mu$ on γ, we see that,

$$V_1^\mu(s,\omega) = \frac{e^{-\widetilde{\nu}s}}{2i\pi}\int_{\gamma(s)} \frac{e^{\widetilde{\nu}s'}\widetilde{V}_1(s',\omega)}{s-s'}ds' + \mathcal{O}(e^{-(\nu-\widetilde{\nu})/(2\mu)-\widetilde{\nu}\mathrm{Re}\, s}),$$

where,

$$\gamma(s) := \left(\gamma \cap \{\text{Re } s' \leq \text{Re } s + \frac{1}{\mu}\}\right) \cup (\text{Re } s + \frac{1}{\mu} + 2i[-\mu, \mu]).$$

In particular, $\gamma(s)$ is a simple oriented loop around s, and therefore, one obtains,

$$\begin{aligned} V_1^\mu(s,\omega) - \widetilde{V}_1(s,\omega) &= \frac{e^{-\widetilde{\nu}s}}{2i\pi}\int_{\gamma(s)} \frac{e^{\widetilde{\nu}s'}\widetilde{V}_1(s',\omega) - e^{\widetilde{\nu}s}\widetilde{V}_1(s,\omega)}{s-s'}ds' \\ &\quad + \mathcal{O}(e^{-(\nu-\widetilde{\nu})/(2\mu)-\widetilde{\nu}\text{Re } s}). \end{aligned} \tag{4.6}$$

Then, writing,

$$e^{\widetilde{\nu}s'}\widetilde{V}_1(s',\omega) - e^{\widetilde{\nu}s}\widetilde{V}_1(s,\omega) = (s-s')f(s,s',\omega) + \overline{(s-s')}g(s,s',\omega), \tag{4.7}$$

with $|\overline{\partial}_{s'}f| + |g| = \mathcal{O}(\mu^\infty)$, by Stokes' formula, we see that, for Re $s \leq 2/\mu$ and $|\text{Im } s| \leq \mu$, we have,

$$V_1^\mu(s,\omega) - \widetilde{V}_1(s,\omega) = \mathcal{O}(\mu^\infty e^{-\widetilde{\nu}\text{Re } s}).$$

When Re $s > 2/\mu$ and $|\text{Im } s| \leq \mu$, setting,

$$\gamma_1(s) := \left(\gamma \cap \{\text{Re } s' \leq \frac{1}{\mu}\}\right) \cup (\frac{1}{\mu} + 2i[-\mu, \mu]),$$

Stokes' formula directly gives,

$$\int_{\gamma_1(s)} \frac{e^{\widetilde{\nu}s'}\widetilde{V}_1(s',\omega)}{s-s'}ds' = \mathcal{O}(\mu^\infty),$$

and thus, using again that $\widetilde{V}_1(s',\omega) = \mathcal{O}(e^{-\nu\text{Re } s'})$, in that case we obtain,

$$|V_1^\mu(s,\omega)| + |\widetilde{V}_1(s,\omega)| = \mathcal{O}(\mu^\infty e^{-\widetilde{\nu}\text{Re } s}).$$

In particular, in both cases we obtain,

$$V_1^\mu(s,\omega) - \widetilde{V}_1(s,\omega) = \mathcal{O}(\mu^\infty e^{-\widetilde{\nu}\text{Re } s}), \tag{4.8}$$

uniformly for Re $s \geq -\mu$, $|\text{Im } s| \leq \mu$ and $\mu > 0$ small enough.

Then, for $\alpha \in \mathbb{N}^n$ arbitrary, by differentiating (4.4) and observing that,

$$\begin{aligned} e^{\widetilde{\nu}s'}\widetilde{V}_1(s',\omega) - \sum_{k=0}^{N} \frac{1}{k!}(s'-s)^k\partial_s^k\left(e^{\widetilde{\nu}s}\widetilde{V}_1(s,\omega)\right) \\ = (s'-s)^{N+1}f_N(s,s',\omega) + g_N(s,s',\omega), \end{aligned}$$

with $|\overline{\partial}_{s'}f_N| + |g_N| = \mathcal{O}(\mu^\infty)$, the same procedure gives,

$$\partial^\alpha(V_1^\mu(s,\omega) - \widetilde{V}_1(s,\omega)) = \mathcal{O}(\mu^\infty e^{-\widetilde{\nu}\text{Re } s}), \tag{4.9}$$

uniformly for Re $s \geq -\mu$, $|\text{Im } s| \leq \mu$ and $\mu > 0$ small enough. In particular, using the properties of $\widetilde{V}_1$, on the same set we also obtain,

$$\partial^\alpha V_1^\mu(s,\omega) = \mathcal{O}(e^{-\widetilde{\nu}\text{Re } s}), \tag{4.10}$$

uniformly.

Now, let $\chi_1 \in C^\infty(\mathbb{R}; [0,1])$ be such that $\chi_1 = 1$ on $(-\infty, -1]$, and $\chi_1 = 0$ on $\mathbb{R}_+$. We set,

$$V_2^\mu(s,\omega) := \chi_1(s/\mu)\widetilde{V}_1(s,\omega) + (1-\chi_1(s/\mu))V_1^\mu(s,\omega). \tag{4.11}$$

In particular, V_2^μ is well defined and smooth on $\mathbb{R}_- \cup (\mathbb{R}_+ + i[-\mu,\mu])$, and one has,

$$\begin{aligned} &V_2^\mu = \widetilde{V}_1 \text{ for } s \in (-\infty, -\mu];\\ &V_2^\mu = V_1^\mu \text{ for } s \in \mathbb{R}_+ + i[-\mu,\mu];\\ &\partial^\alpha(V_2^\mu - \widetilde{V}_1) = \mathcal{O}(\mu^\infty) \text{ for } s \in [-\mu,\mu]. \end{aligned}$$

Finally, setting,

$$V^\mu(x) := V_2^\mu(\ln|x|, \frac{x}{|x|}), \tag{4.12}$$

for $x \neq 0$, and $V^\mu(0) = \widetilde{V}(0)$, we easily deduce from the previous discussion (in particular (4.8), (4.9) and (4.10), and the fact that $\partial_r = r^{-1}\partial_s$), that V^μ is a $|x|$-analytic $(\mu, \widetilde{\nu})$-approximation of V. □

5 The analytic distortion

In this section, for any $\theta > 0$ small enough, we construct a suitable distortion $x \mapsto x + i\theta A(x)$ verifying $A(x) = x$ for $|x|$ large enough, and such that, for $\mu \geq \theta$, the resolvent $(P_\theta^\mu - z)^{-1}$ of the corresponding distorted Hamiltonian P_θ^μ, admits sufficiently good estimates when Im $z \geq h^{n_1}\theta$.

We fix $R_0 \geq 1$ arbitrarily, and we have,

Lemma 5.1. *For any $\lambda > 1$ large enough, there exists $f_\lambda \in C^\infty(\mathbb{R}_+)$, such that,*

(i) $\mathrm{Supp}\, f_\lambda \subset [R_0, +\infty)$*;*

(ii) $f_\lambda(r) = \lambda r$ *for* $r \geq 2\ln\lambda$*;*

(iii) $0 \leq f_\lambda(r) \leq r f_\lambda'(r) \leq 2\lambda r$ *everywhere;*

(iv) $f_\lambda' + |f_\lambda''| = \mathcal{O}(1 + f_\lambda)$ *uniformly;*

(v) For any $k \geq 1$, $f_\lambda^{(k)} = \mathcal{O}(\lambda)$ *uniformly.*

The construction of such an f_λ is made in Appendix A.1.

Now, we take $\lambda := h^{-n_1}$, and we set,

$$b(r) := \frac{1}{\lambda} f_\lambda(r). \tag{5.1}$$

By the previous lemma, b verifies,

- $\mathrm{Supp}\, b \subset [R_0, +\infty)$;

- $b(r) = r$ for $r \geq 2n_1 \ln \frac{1}{h}$;
- $0 \leq b(r) \leq rb'(r) \leq 2r$ everywhere;
- $b' + |b''| = \mathcal{O}(h^{n_1} + b)$ uniformly;
- For any $k \geq 1$, $b^{(k)} = \mathcal{O}(1)$ uniformly.

We set,

$$A(x) := b(|x|)\frac{x}{|x|} = a(|x|)x,$$

where $a(r) := r^{-1}b(r) \in C^\infty(\mathbb{R}_+)$. For $\mu \geq \theta$ (both small enough), we can define the distorted operator P_θ^μ as in (2.4) obtained from P^μ by using the distortion,

$$\Phi_\theta \, : \, \mathbb{R}^n \ni x \mapsto x + i\theta A(x) \in \mathbb{C}^n. \tag{5.2}$$

Here we use the fact that $|A(x)| \leq 2|x|$, and we also observe that, for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq 1$, one has $\partial^\alpha \Phi_\theta(x) = \mathcal{O}(\theta\langle x\rangle^{1-|\alpha|})$ uniformly.

Proposition 5.2. *If R_0 is fixed sufficiently large, then, for $0 < \theta \leq \mu$ both small enough, $h > 0$ small enough, $u \in H^2(\mathbb{R}^n)$, and $z \in \mathbb{C}$ such that* $\mathrm{Re}\, z \in [\lambda_0/2, 2\lambda_0]$ *and* $\mathrm{Im}\, z \geq h^{n_1}\theta$, *one has,*

$$|\langle (P_\theta^\mu - z)u, u\rangle_{L^2}| \geq \frac{\mathrm{Im}\, z}{2}\|u\|_{L^2}^2.$$

Proof. Setting $F := {}^t dA(x) = dA(x)$, and $V_\theta^\mu(x) := V^\mu(x + i\theta A(x))$, we have,

$$\begin{aligned}
\langle P_\theta^\mu u, u\rangle &= \langle [(I + i\theta F(x))^{-1}hD_x]^2 u, u\rangle + \langle V_\theta^\mu u, u\rangle \\
&= \langle (1 + i\theta F(x))^{-2}hD_x u, hD_x u\rangle \\
&\quad + ih\langle [({}^t\nabla_x)(I + i\theta F(x))^{-1}](I + i\theta F(x))^{-1}h\nabla_x u, u\rangle + \langle V_\theta^\mu u, u\rangle.
\end{aligned}$$

Therefore, using Lemma A.1, and the fact that, for complex x, we have,

$$|\mathrm{Im}\, V^\mu(x)| = \mathcal{O}(|\mathrm{Im}\, x|\langle \mathrm{Re}\, x\rangle^{-\nu-1}),$$

we find,

$$\begin{aligned}
\mathrm{Im}\, \langle P_\theta^\mu u, u\rangle &\leq -\theta\|\sqrt{a(|x|)}\, hD_x u\|^2 \\
&\quad + Ch\theta \int \left(|b''| + \frac{b'}{|x|} + \frac{b}{|x|^2}\right)|hD_x u| \cdot |u| dx + C_0\theta \left\| \frac{\sqrt{b}}{|x|^{\frac{\nu+1}{2}}} u \right\|^2
\end{aligned}$$

for some constants $C, C_0 > 0$, C_0 independent of the choice of R_0.

Thus, using the properties of b after Lemma 5.1, we obtain (with some other constant $C > 0$),

$$\begin{aligned}
\mathrm{Im}\, \langle P_\theta^\mu u, u\rangle &\leq -\theta\|\sqrt{a(|x|)}hD_x u\|^2 \\
&\quad + Ch\theta \int \left(|b''| + \frac{b}{|x|} + h^{n_1}\right)|hD_x u| \cdot |u| dx + C_0 R_0^{-\nu}\theta \left\|\sqrt{a}u\right\|^2.
\end{aligned} \tag{5.3}$$

On the other hand, for $z\in\mathbb{C}$, a similar computation gives,

$$\begin{aligned}
\mathrm{Re}\,\langle\sqrt{a}(P_\theta^\mu - z)u, \sqrt{a}u\rangle &= -(\mathrm{Re}\, z)\|\sqrt{a}u\|^2\\
&\quad +\mathrm{Re}\,\langle\sqrt{a}[(I+i\theta F(x))^{-1}hD_x]^2u, \sqrt{a}u\rangle\\
&\quad +\mathrm{Re}\,\langle\sqrt{a}V_\theta^\mu u, \sqrt{a}u\rangle,\\
&\le -(\mathrm{Re}\, z)\|\sqrt{a}u\|^2 + (1-2\theta)^{-2}\|\sqrt{a}hD_xu\|^2\\
&\quad +Ch\int\left(|b''|+\frac{b}{|x|}+h^{n_1}\right)|hD_xu|\cdot|u|dx\\
&\quad +C_0R_0^{-\nu}\left\|\sqrt{a}u\right\|^2,
\end{aligned}$$

still with C, C_0 positive constants, and C_0 independent of the choice of R_0. Therefore, if $\mathrm{Re}\, z\ge\lambda_0/2>0$ and R_0 is chosen sufficiently large, then, for θ small enough, we obtain,

$$\begin{aligned}
\|\sqrt{a}u\|^2 &\le 4\lambda_0^{-1}\|\sqrt{a}hD_xu\|^2\\
&\quad +4C\lambda_0^{-1}h\int\left(|b''|+\frac{b}{|x|}+h^{n_1}\right)|hD_xu|\cdot|u|dx\\
&\quad +4\lambda_0^{-1}|\langle\sqrt{a}(P_\theta^\mu - z)u, \sqrt{a}u\rangle|.
\end{aligned}\tag{5.4}$$

The insertion of this estimate into (5.3) gives,

$$\begin{aligned}
\mathrm{Im}\,\langle P_\theta^\mu u, u\rangle &\le -(1-4C_0\lambda_0^{-1}R_0^{-\nu})\theta\|\sqrt{a}hD_xu\|^2\\
&\quad +C'h\theta\int\left(|b''|+\frac{b}{|x|}+h^{n_1}\right)|hD_xu|\cdot|u|dx\\
&\quad +C'\theta|\langle\sqrt{a}(P_\theta^\mu - z)u, \sqrt{a}u\rangle|,
\end{aligned}\tag{5.5}$$

with $C'>0$ a constant.

Now, for $r\ge 2n_1\ln\frac{1}{h}$, by construction we have $b''(r)=0$, while, for $r\le 2n_1\ln\frac{1}{h}$, we have,

$$|b''(r)| = \mathcal{O}(h^{n_1}+b) = \mathcal{O}(h^{n_1}+(\ln\frac{1}{h})a).\tag{5.6}$$

Then, we deduce from (5.5),

$$\begin{aligned}
\mathrm{Im}\,\langle P_\theta^\mu u, u\rangle &\le -(1-4C_0\lambda_0^{-1}R_0^{-\nu})\theta\|\sqrt{a}hD_xu\|^2\\
&\quad +C'h\theta\ln\frac{1}{h}\|\sqrt{a}hD_xu\|\cdot\|\sqrt{a}u\|\\
&\quad +C'h^{n_1+1}\theta\|hD_xu\|\cdot\|u\| + C'\theta|\langle\sqrt{a}(P_\theta^\mu - z)u, \sqrt{a}u\rangle|,
\end{aligned}\tag{5.7}$$

with some other constant $C'>0$. Using again (5.6), we also deduce from (5.4),

$$\begin{aligned}
\|\sqrt{a}u\|^2 &= \mathcal{O}(\|\sqrt{a}hD_xu\|^2 + |\langle\sqrt{a}(P_\theta^\mu - z)u, \sqrt{a}u\rangle|\\
&\qquad +h^{n_1+1}\|hD_xu\|\cdot\|u\|),
\end{aligned}$$

uniformly for $h>0$ small enough, and thus, by (5.7),

$$\begin{aligned}
\mathrm{Im}\,\langle P_\theta^\mu u, u\rangle &\le -(1-4C_0\lambda_0^{-1}R_0^{-\nu}-Ch\ln\frac{1}{h})\theta\|\sqrt{a}hD_xu\|^2\\
&\quad +Ch^{n_1+1}\theta\|hD_xu\|\cdot\|u\| + C\theta|\langle\sqrt{a}(P_\theta^\mu - z)u, \sqrt{a}u\rangle|.
\end{aligned}\tag{5.8}$$

Finally, for $\mathrm{Re}\, z \leq 2\lambda_0$, we use the (standard) ellipticity of the second-order partial differential operator $\mathrm{Re}\, P_\theta^\mu$, and the properties of V^μ, to obtain,

$$\mathrm{Re}\, \langle (P_\theta^\mu - z)u, u\rangle \geq \frac{1}{C}\|hD_x u\|^2 - C\|u\|^2,$$

where C is again a new positive constant, independent of μ and θ. Combining with (5.8), and possibly increasing the value of R_0, this leads to,

$$\begin{aligned} \mathrm{Im}\, \langle (P_\theta^\mu - z)u, u\rangle \;\leq\; & (Ch^{n_1+1}\theta - \mathrm{Im}\, z)\|u\|^2 \\ & + Ch^{n_1+1}\theta |\langle (P_\theta^\mu - z)u, u\rangle|^{\frac{1}{2}}\|u\| + C\theta|\langle (P_\theta^\mu - z)u, u\rangle|, \end{aligned} \tag{5.9}$$

and thus, for $\mathrm{Im}\, z \geq h^{n_1}\theta$, and for $h, \theta > 0$ small enough, we can deduce,

$$|\langle (P_\theta^\mu - z)u, u\rangle| \geq \frac{3\mathrm{Im}\, z}{4}\|u\|^2 - Ch^{n_1+1}\theta|\langle (P_\theta^\mu - z)u, u\rangle|^{\frac{1}{2}}\|u\|. \tag{5.10}$$

Then, the result easily follows by solving this second-order inequation where the unkonwn variable is $|\langle (P_\theta^\mu - z)u, u\rangle|^{\frac{1}{2}}$, and by using again that $\mathrm{Im}\, z \gg h^{n_1+1}\theta$. □

6 Proof of Theorem 2.1

6.1 The invertible reference operator

The purpose of this section is to introduce an operator without eigenvalues near λ_0, obtained as a finite-rank perturbation of P_θ^μ, $0 < \theta \leq \mu$, and for which we have a nice estimate for the resolvent in the lower half plane. This operator will be used in the next section to construct a convenient Grushin problem.

Let $\chi_0 \in C_0^\infty(\mathbb{R}_+; [0,1])$ be equal to 1 on $[0, 1 + 2\lambda_0 + \sup|V|]$, and let $C_0 > \sup|\nabla V|$. We set,

$$\begin{aligned} & R = R(h) := 2n_1 \ln \frac{1}{h}; \\ & \widetilde{P}_\theta^\mu := P_\theta^\mu - iC_0\theta\chi_0(h^2D_x^2 + R^{-2}x^2). \end{aligned}$$

Observe that $h^2D_x^2 + R^{-2}x^2$ is unitarily equivalent to $hR^{-1}(D_x^2 + x^2)$, and therefore the rank of $\chi_0(h^2D_x^2 + R^{-2}x^2)$ is $\mathcal{O}(R^n h^{-n})$.

For $m \in \mathbb{R}$, we denote by $S(\langle\xi\rangle^m)$ the set of functions $a \in C^\infty(\mathbb{R}^{2n})$ such that, for all $\alpha \in \mathbb{N}^{2n}$, one has,

$$\partial_{x,\xi}^\alpha a(x,\xi) = \mathcal{O}(\langle\xi\rangle^m) \text{ uniformly}.$$

We also denote

$$\mathrm{Op}_h^W(a)u(x) = \frac{1}{(2\pi h)^n}\iint e^{i(x-y)\xi/h} a\Big(\frac{x+y}{2}, \xi\Big)u(y)\,dy\,d\xi, \tag{6.1}$$

the semiclassical Weyl quantization of such a symbol a.

Denoting by $\widetilde{p}_\theta^\mu \in S(\langle\xi\rangle^2)$ the Weyl symbol of $\widetilde{P}_\theta^\mu$, we see that,

$$\widetilde{p}_\theta^\mu(x,\xi) = [({}^t d\Phi_\theta(x))^{-1}\xi]^2 + V^\mu(\Phi_\theta(x)) - iC_0\theta\chi_0(\xi^2 + R^{-2}x^2) + \mathcal{O}(h\theta\langle\xi\rangle), \tag{6.2}$$

uniformly with respect to (x,ξ), μ, θ, and h, and where the estimate on the remainder is in the sense of symbols (that is, one has the same estimate for all the derivatives). In particular, we have,

$$\mathrm{Re}\, \widetilde{p}_\theta^\mu(x,\xi) = \xi^2 + V(x) + \mathcal{O}(\theta\langle\xi\rangle^2). \tag{6.3}$$

Moreover

- If $|x| \geq R$ and $|\xi|^2 \geq \lambda_0/2$, then,

$$\mathrm{Im}\, \widetilde{p}_\theta^\mu(x,\xi) \leq -\frac{\theta}{C}\langle\xi\rangle^2 + \mathcal{O}(\theta R^{-\nu}) \leq -\frac{\theta}{2C}\langle\xi\rangle^2; \tag{6.4}$$

- If $|x| \leq R$ and $|\xi|^2 \leq 2\lambda_0 + \sup|V|$, then,

$$\mathrm{Im}\, \widetilde{p}_\theta^\mu \leq -C_0\theta + \theta\sup|\nabla V| + \mathcal{O}(h\theta) \leq -\frac{\theta}{2C}, \tag{6.5}$$

where $C > 0$ is a constant, and the estimates are valid for h small enough. (For (6.5), we have used the fact that $\mathrm{Im}\, [({}^t d\Phi_\theta(x))^{-1}\xi]^2 \leq 0$, that is due to the particular form of $\Phi_\theta(x)$. See Lemma A.1 in appendix.)

We have,

Proposition 6.1. *There exists a constant $\widetilde{C} \geq 1$ such that, for all $\mu > 0$, for all $\theta \in (0,\mu]$, for all z verifying $|\mathrm{Re}\, z - \lambda_0| + \theta^{-1}|\mathrm{Im}\, z| \leq \frac{4}{\widetilde{C}}$, and for all $h \in (0, 1/\widetilde{C}]$, one has,*

$$\|(z - \widetilde{P}_\theta^\mu)^{-1}\| \leq \widetilde{C}\theta^{-1}.$$

Proof. We take two functions $\varphi_1, \varphi_2 \in C_b^\infty(\mathbb{R}^{2n}; [0,1])$ (the space of smooth functions bounded with all their derivatives), such that,

- $\varphi_1^2 + \varphi_2^2 = 1$ on $\mathbb{R}^{2n}$;
- $\mathrm{Supp}\,\varphi_1$ is included in a small enough neighborhood of $\{\xi^2 + V(x) = \lambda_0\}$;
- $\varphi_1 = 1$ near $\{\xi^2 + V(x) = \lambda_0\}$.

In particular, φ_1 can be chosen in such a way that, on $\mathrm{Supp}\,\varphi_1$, one has either $|x| \geq R$ together with $|\xi|^2 \geq \lambda_0/2$, or $|x| \leq R$ together with $|\xi|^2 \leq 2\lambda_0 + \sup|V|$. Therefore, we deduce from (6.4)-(6.5),

$$\frac{1}{\theta}\mathrm{Im}\, \widetilde{p}_\theta^\mu \leq -\frac{1}{2C} \text{ on } \mathrm{Supp}\,\varphi_1,$$

and thus,

$$\varphi_1^2\frac{1}{\theta}\mathrm{Im}\, \widetilde{p}_\theta^\mu + \frac{1}{2C}\varphi_1^2 \leq 0 \text{ on } \mathbb{R}^{2n}. \tag{6.6}$$

Moreover, it is easy to check that the function $(x,\xi)\mapsto \theta^{-1}\mathrm{Im}\,\widetilde{p}_\theta^\mu$ is a nice symbol in $S(\langle\xi\rangle^2)$, uniformly with respect to μ and θ, that is, for all $\alpha\in\mathbb{N}^{2n}$, one has,

$$\partial_{x,\xi}^\alpha(\theta^{-1}\mathrm{Im}\,\widetilde{p}_\theta^\mu)(x,\xi)=\mathcal{O}(\langle\xi\rangle^2)\text{ uniformly},$$

and we see from (6.2), that,

$$\theta^{-1}\mathrm{Im}\,\widetilde{p}_\theta^\mu\leq CR^{-\nu}+Ch\langle\xi\rangle,$$

with some new uniform constant $C>0$.

Then, setting $\phi_j:=\mathrm{Op}_h^W(\varphi_j)$, writing $I=\phi_1^2u+\phi_2^2u+hQ$ where Q is a uniformly bounded pseudodifferential operator, and using the sharp Gårding inequality, we obtain,

$$\begin{aligned}\langle\theta^{-1}\mathrm{Im}\,\widetilde{P}_\theta^\mu u,u\rangle &= \langle\phi_1\theta^{-1}\mathrm{Im}\,\widetilde{P}_\theta^\mu\phi_1u,u\rangle+\langle\theta^{-1}\mathrm{Im}\,\widetilde{P}_\theta^\mu\phi_2u,\phi_2u\rangle+\mathcal{O}(h\|u\|_{H^1}^2)\\ &\leq -\frac{1}{2C}\|\phi_1u\|^2+CR^{-\nu}\|\phi_2u\|^2+Ch\|\langle hD_x\rangle u\|^2,\end{aligned}$$

for all $u\in H^2(\mathbb{R}^n)$, and still for some new uniform constant $C>0$. Hence,

$$|\mathrm{Im}\,\langle\widetilde{P}_\theta^\mu u,u\rangle|\geq\frac{\theta}{2C}\|\phi_1u\|^2-C\theta R^{-\nu}\|\phi_2u\|^2-Ch\theta\|\langle hD_x\rangle u\|^2. \tag{6.7}$$

On the other hand, since $\mathrm{Re}\,\widetilde{p}_\theta^\mu-\lambda_0\in S(\langle\xi\rangle^2)$ is uniformly elliptic on $\mathrm{Supp}\,\varphi_2$, the symbolic calculus permits us to construct $a\in S(\langle\xi\rangle^{-2})$ (still depending on μ and θ, but with uniform estimates), such that,

$$a\sharp(\widetilde{p}_{k,\theta}-\lambda_0)=\varphi_2\sharp\varphi_2+\mathcal{O}(h^\infty)\text{ in }S(1),$$

where $\sharp$ stands for the Weyl composition of symbols. As a consequence, denoting by A the Weyl quantization of a, we obtain,

$$\|\langle hD_x\rangle\phi_2u\|^2=\langle\langle hD_x\rangle^2A(\widetilde{P}_\theta^\mu-\lambda_0)u,u\rangle+\mathcal{O}(h)\|u\|^2,$$

and thus,

$$\|(\widetilde{P}_\theta^\mu-\lambda_0)u\|\cdot\|u\|\geq\frac{1}{C}\|\langle hD_x\rangle\phi_2u\|^2-Ch\|u\|^2. \tag{6.8}$$

Now, if $z\in\mathbb{C}$ is such that $|\mathrm{Re}\,z-\lambda_0|\leq\varepsilon$ and $|\mathrm{Im}\,z|\leq\varepsilon\theta$ ($\varepsilon>0$ fixed), we deduce from (6.7)-(6.8),

$$\begin{aligned}\|(\widetilde{P}_\theta^\mu-z)u\|\cdot\|u\|\geq|\mathrm{Im}\,\langle(\widetilde{P}_\theta^\mu-z)u,u\rangle|&\geq\frac{\theta}{2C}\|\phi_1u\|^2-C\theta R^{-\nu}\|\phi_2u\|^2\\ &\quad -Ch\theta\|\langle hD_x\rangle u\|^2-\varepsilon\theta\|u\|^2;\\ \theta\|(\widetilde{P}_\theta^\mu-z)u\|\cdot\|u\|&\geq\frac{\theta}{C}\|\langle hD_x\rangle\phi_2u\|^2-Ch\theta\|u\|^2-2\varepsilon\theta\|u\|^2,\end{aligned}$$

that yields

$$\begin{aligned}(1+\theta)\|(\widetilde{P}_\theta^\mu-z)u\|\cdot\|u\| &\geq \frac{\theta}{2C}\left(\|\phi_1u\|^2+\|\langle hD_x\rangle\phi_2u\|^2\right)\\ &\quad -\theta(2Ch+CR^{-\nu}+3\varepsilon)\|\langle hD_x\rangle u\|^2.\end{aligned} \tag{6.9}$$

Moreover, since ξ remains bounded on $\mathrm{Supp}\,\varphi_1$, we see that the norms $\|\langle hD_x\rangle u\|$ and $\|\phi_1u\|+\|\langle hD_x\rangle\phi_2u\|$ are uniformly equivalent, and thus, for ε and h small enough, we deduce from (6.9),

$$\|(\widetilde{P}_\theta^\mu-z)u\|\cdot\|u\|\geq\frac{\theta}{4C}\|\langle hD_x\rangle u\|^2,$$

and the result follows. □

6.2 The Grushin problem

In this section, we reduce the estimate on $(P_\theta^\mu - z)^{-1}$ in Theorem 2.1, to that of a finite matrix, by means of some convenient Grushin problem.

Denote by $(e_1, \dots, e_M)$ an orthonormal basis of the range of the operator,

$$K := C_0\chi_0(h^2D_x^2 + R^{-2}x^2).$$

In particular, $M = M(h)$ verifies,

$$M = \mathcal{O}(R^n h^{-n}). \tag{6.10}$$

Let $z \in \mathbb{C}$, and consider the two operators,

$$R_+ : \begin{array}{ccc} L^2(\mathbb{R}^n) & \to & \mathbb{C}^M \\ u & \mapsto & (\langle u, e_j\rangle)_{1\le j\le M}, \end{array}$$

and,

$$R_-(z) : \begin{array}{ccc} \mathbb{C}^M & \to & L^2(\mathbb{R}^n) \\ u^- & \mapsto & \sum_{j=1}^M u_j^-(\widetilde{P}_\theta^\mu - z)e_j. \end{array}$$

Then, the Grushin operator,

$$\mathcal{G}(z) := \begin{pmatrix} P_\theta^\mu - z & R_-(z) \\ R_+ & 0 \end{pmatrix},$$

is well defined from $H^2(\mathbb{R}^n)\times\mathbb{C}^M$ to $L^2(\mathbb{R}^n)\times\mathbb{C}^M$, and for z as in Proposition 6.1, it is easy to show that $\mathcal{G}(z)$ is invertible, and its inverse is given by,

$$\mathcal{G}(z)^{-1} := \begin{pmatrix} E(z) & E^+(z) \\ E^-(z) & E^{-+}(z) \end{pmatrix},$$

where,

$$E(z) = (1 - T_M)(\widetilde{P}_\theta^\mu - z)^{-1}, \text{ with } T_M v := \sum_{j=1}^M \langle v, e_j\rangle e_j \ (v \in L^2);$$

$$E^+(z)v^+ = \sum_{j=1}^M v_j^+(e_j + i\theta(1-T_M)(\widetilde{P}_\theta^\mu - z)^{-1}Ke_j),$$

$$(v_+ = (v_j^+)_{1\le j\le M} \in \mathbb{C}^M);$$

$$E^-(z)v = (\langle(\widetilde{P}_\theta^\mu - z)^{-1}v, e_j\rangle)_{1\le j\le M};$$

$$E^{-+}(z)v^+ = -v^+ + i\theta\left(\sum_{\ell=1}^M v_\ell^+\langle(\widetilde{P}_\theta^\mu - z)^{-1}Ke_\ell, e_j\rangle\right)_{1\le j\le M}.$$

Proposition 6.1 gives

$$\|E(z)\| + \|E^-(z)\| = \mathcal{O}(\theta^{-1}); \tag{6.11}$$

$$\|E^+(z)\| + \|E^{-+}(z)\| = \mathcal{O}(1), \tag{6.12}$$

uniformly for $\mu > 0$, $\theta \in (0, \mu]$, $h > 0$ small enough, and $|\mathrm{Re}\, z - \lambda_0| + \theta^{-1}|\mathrm{Im}\, z|$ small enough.

Hence, using the algebraic identity,

$$(P_\theta^\mu - z)^{-1} = E(z) - E^+(z)E^{-+}(z)^{-1}E^-(z), \tag{6.13}$$

we finally obtain,

Proposition 6.2. *If $z \in \mathbb{C}$ is such that $|\mathrm{Re}\, z - \lambda_0| \le \widetilde{C}^{-1}$ and $|\mathrm{Im}\, z| \le 2\widetilde{C}^{-1}\theta$, and $E^{-+}(z)$ is invertible, then so is $P_\theta^\mu - z$, and one has,*

$$\|(P_\theta^\mu - z)^{-1}\| = \mathcal{O}(\theta^{-1}(1 + \|E^{-+}(z)^{-1}\|)),$$

uniformly with respect to $\mu > 0$, $\theta \in (0, \mu]$, $h > 0$ small enough, and z such that $|\mathrm{Re}\, z - \lambda_0| \le \widetilde{C}^{-1}$ and $|\mathrm{Im}\, z| \le \widetilde{C}^{-1}\theta$.

Therefore, we have reduced the study of $(P_\theta^\mu - z)^{-1}$ to that of the $M \times M$ matrix $E^{-+}(z)^{-1}$.

6.3 Using the Maximum Principle

For $z \in J + i[-\theta/\widetilde{C}, 2\theta/\widetilde{C}]$, we define,

$$D(z) := \det E^{-+}(z).$$

Then, $z \mapsto D(z)$ is holomorphic in $J + i[-\theta/\widetilde{C}, 2\theta/\widetilde{C}]$. Using (6.13) and setting $N := \#(\sigma(P_\theta^\mu) \cap (J + i[-\theta/\widetilde{C}, 2\theta/\widetilde{C}])$, we see that $D(z)$ can be written as,

$$D(z) = G(z)\prod_{\ell=1}^{N}(z - \rho_\ell),$$

with G holomorphic in $J + i[-\theta/\widetilde{C}, 2\theta/\widetilde{C}]$, $G(z) \ne 0$ for all $z \in J - i[0, \widetilde{C}^{-1}\theta]$.

Moreover, using (6.12) and (6.10), we obtain,

$$|D(z)| = \prod_{\lambda \in \sigma(E^{-+}(z))} |\lambda| \le \|E^{-+}(z)\|^M \le C_1 e^{C_1 R^n h^{-n}}, \tag{6.14}$$

for some uniform constant $C_1 > 0$.

Lemma 6.3. *For every $\theta \in [0, \mu]$, there exists $r_\theta \in [\theta/(2\widetilde{C}), \theta/\widetilde{C}]$, such that for all $z \in J - ir_\theta$, and for all $\ell = 1, \dots, N$, one has,*

$$|z - \rho_\ell| \ge \frac{\theta}{8\widetilde{C}N}.$$

Proof. By contradiction, if it was not the case, then for all t in $[-\theta/2\widetilde{C}, -\theta/\widetilde{C}]$, there should exist ℓ such that,

$$|t - \mathrm{Im}\, \rho_\ell| < \frac{\theta}{8\widetilde{C}N}.$$

Therefore, the interval $[-\theta/2\widetilde{C}, -\theta/\widetilde{C}]$ would be included in $\cup_{\ell=1}^{N}[\mathrm{Im}\, \rho_\ell - \theta/(8\widetilde{C}N), \mathrm{Im}\, \rho_\ell + \theta/(8\widetilde{C}N)]$, which is impossible because of their respective size. $\square$

From now on, we assume $\mathcal{P}(\widetilde{\mu}, \mu; I, J)$ and setting,

$$\mathcal{W}_\theta(J) := J + i[-r_\theta, 2\theta/\widetilde{C}],$$

we deduce from Lemma 6.3 that, for $\theta \in (0, \widetilde{\mu}]$, z on the boundary of $\mathcal{W}_\theta(J)$, and for all $\ell = 1, \dots, N$, we have,

$$|z - \rho_\ell| \geq \frac{1}{C_2}\theta,$$

for some constant $C_2 > 0$. As a consequence, using (6.14), on this set we obtain,

$$|G(z)| \leq \theta^{-C_3} e^{C_3 R^n h^{-n}},$$

with some other uniform constant $C_3 > 0$. Then, the maximum principle tells us that this estimate remains valid in the interior of $\mathcal{W}_\theta(J)$, that is,

Proposition 6.4. *There exists a constant $C_3 > 0$ such that, for all $\widetilde{\mu}$, μ, I and J verfying (2.6) – (2.7) such that $\mathcal{P}(\widetilde{\mu}, \mu; I, J)$ holds, one has,*

$$|G(z)| \leq \theta^{-C_3} e^{C_3 R^n h^{-n}},$$

for all $\theta \in (0, \widetilde{\mu}]$, $z \in \mathcal{W}_\theta(J)$, and $h \in (0, 1/C_3]$.

6.4 Using the Harnack Inequality

Since $G(z) \neq 0$ on $\mathcal{W}_\theta(J)$, we can consider the function,

$$H(z) := C_3 R^n h^{-n} - C_3 \ln\theta - \ln|G(z)|.$$

Then, H is harmonic in $\mathcal{W}_\theta(J)$, and, by Proposition 6.4, it is also nonnegative.

Using the algebraic formula,

$$E^{-+}(z)^{-1} = -R_+(P_\theta^\mu - z)^{-1} R_-(z),$$

and the fact that $(P_\theta^\mu - z)^{-1} R_-(z) u^- = \sum_{j=1}^M u_j (I - i\theta(P_\theta^\mu - z)^{-1} K) e_j$, we deduce from Proposition 5.2 that, for $z \in [\lambda_0/2, 2\lambda_0] + i[\theta h^{n_1}, 1]$, one has,

$$\|E^{-+}(z)^{-1}\| \leq 1 + 2C_0 h^{-n_1}.$$

As a consequence, for such values of z, we obtain,

$$\frac{1}{D(z)} = \det E^{-+}(z)^{-1} \leq (1 + 2C_0 h^{-n_1})^M,$$

and thus,

$$|G(z)| = |D(z)| \prod_{\ell=1}^N |z - \rho_\ell|^{-1} \geq \frac{1}{C_4} h^{n_1 M},$$

with some uniform constant $C_4 > 0$. In particular, for any $\lambda \in \mathbb{R}$ such that $\lambda + i\theta h^{n_1} \in \mathcal{W}_\theta(J)$, this gives,

$$H(\lambda + i\theta h^{n_1}) \leq C_3 R^n h^{-n} - C_3 \ln\theta + \ln C_4 - n_1 M \ln h. \tag{6.15}$$

Now, the Harnack inequality tells us that, for any λ, r, such that,

$$\operatorname{dist}(\lambda, \mathbb{R} \setminus J) \geq \widetilde{C}^{-1}\theta \quad ; \quad r \in [0, \widetilde{C}^{-1}\theta)$$

and for any $\alpha \in \mathbb{R}$, one has,

$$H(\lambda + ih^{n_1}\theta + re^{i\alpha}) \leq \frac{\widetilde{C}^{-2}\theta^2}{(\widetilde{C}^{-1}\theta - r)^2} H(\lambda + ih^{n_1}\theta).$$

In particular, setting

$$\widetilde{\mathcal{W}_\theta}(J) := \left\{ z \in \mathbb{C} \,;\, \operatorname{dist}(\operatorname{Re} z, \mathbb{R} \setminus J) \geq \widetilde{C}^{-1}\theta \,, |\operatorname{Im} z| \leq (2\widetilde{C})^{-1}\theta \right\},$$

and using (6.15), we find,

$$H(z) \leq 5C_3R^nh^{-n} - 5C_3 \ln\theta + 5\ln C_4 - 5n_1M\ln h,$$

for all $z \in \widetilde{\mathcal{W}_\theta}(J)$, that is,

$$\ln|G(z)| \geq -4C_3R^nh^{-n} + 4C_3\ln\theta - 5\ln C_4 + 5n_1M\ln h,$$

or, equivalently,

$$|G(z)| \geq C_4^{-5}\theta^{4C_3}h^{5n_1M}e^{-4C_3R^nh^{-n}}. \tag{6.16}$$

Finally, writing $E^{-+}(z)^{-1} = D(z)^{-1}\widetilde{E}^{-+}(z)$, where $\widetilde{E}^{-+}(z)$ stands for the transposed of the comatrix of $E^{-+}(z)$, we see that,

$$\|E^{-+}(z)^{-1}\| \leq e^{CM}|G(z)|^{-1}\prod_{\ell=1}^{N}|z-\rho_\ell|^{-1},$$

and therefore, we deduce from (6.16) and (6.10),

$$\|E^{-+}(z)^{-1}\| \leq \theta^{-C}h^{-CR^nh^{-n}}\prod_{\ell=1}^{N}|z-\rho_\ell|^{-1},$$

with some new uniform constant $C \geq 1$. Thus, using Proposition 6.2, and the fact that $R = \mathcal{O}(|\ln h|)$, we have proved,

Proposition 6.5. *There exists a constant $\check{C} > 0$ such that, for all $\widetilde{\mu}$, μ, I and J verifying (2.6) – (2.7) such that $\mathcal{P}(\widetilde{\mu}, \mu; I, J)$ holds, one has,*

$$\|(P_\theta^\mu - z)^{-1}\| \leq \theta^{-\check{C}}h^{-\check{C}|\ln h|^nh^{-n}}\prod_{\ell=1}^{N}|z-\rho_\ell|^{-1},$$

for all $\theta \in (0, \widetilde{\mu}]$, $z \in \widetilde{\mathcal{W}_\theta}(J)$, and $h \in (0, 1/\check{C}]$.

6.5 Using the 3-lines theorem

Now, following an idea of [22], we define,

$$\Psi(z) := \int_a^b e^{-(z-\lambda)^2/\theta^2} d\lambda,$$

where,

$$[a,b] := \{\lambda \in \mathbb{R} \,;\, \mathrm{dist}\,(\lambda, \mathbb{R}\setminus J) \geq \widetilde{C}^{-1}\theta + \check{C}^{1/2}\omega_h(\theta)\}.$$

We have,

- If $\mathrm{Im}\, z = 2\theta h^{n_1}$, then,

$$|\Psi(z)| \leq (b-a)e^{4h^{2n_1}} = \mathcal{O}(h^\delta) \leq 1;$$

- If $\mathrm{Im}\, z = -\theta/(2\widetilde{C})$, then,

$$|\Psi(z)| \leq (b-a)e^{1/4\widetilde{C}^2} = \mathcal{O}(h^\delta) \leq 1;$$

- If $\mathrm{Re}\, z \in \left\{a - \check{C}^{1/2}\omega_h(\theta), b + \check{C}^{1/2}\omega_h(\theta)\right\}$ and $\mathrm{Im}\, z \in [-\theta/(2\widetilde{C}), 2\theta h^{n_1}]$, then,

$$|\Psi(z)| \leq (b-a)e^{1/4\widetilde{C}^2}e^{-\check{C}\omega_h(\theta)^2/\theta^2} = \mathcal{O}(h^\delta)\theta^{\check{C}}h^{\check{C}|\ln h|^n h^{-n}} \leq \theta^{\check{C}}h^{\check{C}|\ln h|^n h^{-n}}.$$

Then, for $z \in \widetilde{\mathcal{W}_\theta}(J)$, we consider the operator-valued function,

$$Q(z) := \Psi(z)\prod_{\ell=1}^{N}(z-\rho_\ell)(P_\theta^\mu - z)^{-1},$$

that is holomorphic on $\widetilde{\mathcal{W}_\theta}(J)$ (this can be seen, e.g., from (6.13)). Using, Proposition 5.2, Proposition 6.5, and the previous properties of $\Psi(z)$, we see that, $Q(z)$ verifies,

- If $\mathrm{Im}\, z = 2\theta h^{n_1}$, then,

$$\|Q(z)\| \leq \theta^{-1}h^{-n_1};$$

- If $\mathrm{Im}\, z = -\theta/(2\widetilde{C})$, then,

$$\|Q(z)\| \leq \theta^{-\check{C}}h^{-\check{C}|\ln h|^n h^{-n}};$$

- If $\mathrm{Re}\, z \in \{a - \check{C}^{1/2}\omega_h(\theta), b + \check{C}^{1/2}\omega_h(\theta)\}$ and $\mathrm{Im}\, z \in [-\theta/(2\widetilde{C}), 2\theta h^{n_1}]$, then,

$$\|Q(z)\| \leq 1.$$

Therefore, setting,

$$\check{\mathcal{W}}_\theta(J) := [a - \check{C}^{1/2}\omega_h(\theta), b + \check{C}^{1/2}\omega_h(\theta)] + i[-\theta/(2\widetilde{C}), 2\theta h^{n_1}],$$

(that is included in $\widetilde{\mathcal{W}}_\theta(J)$), we see that the subharmonic function $z \mapsto \ln \|Q(z)\|$ verifies,

$$\ln \|Q(z)\| \leq \psi(z) \ \text{ on } \partial \check{\mathcal{W}}_\theta(J),$$

where ψ is the harmonic function defined by,

$$\begin{aligned}\psi(z): \ &= \ \frac{2\theta h^{n_1} - \mathrm{Im}\, z}{2\theta h^{n_1} + \theta/(2\widetilde{C})}\, \check{C}(|\ln h|^{n+1}h^{-n} + |\ln \theta|)\\ &\quad + \frac{\mathrm{Im}\, z + \theta/(2\widetilde{C})}{2\theta h^{n_1} + \theta/(2\widetilde{C})}\, |\ln(\theta h^{n_1})|.\end{aligned}$$

As a consequence, by the properties of subharmonic functions, we deduce that $\ln \|Q(z)\| \leq \psi(z)$ everywhere in $\check{\mathcal{W}}_\theta(J)$, and in particular, for $|\mathrm{Im}\, z| \leq 2\theta h^{n_1}$, we obtain,

$$\ln \|Q(z)\| \leq 8\widetilde{C}\check{C}h^{n_1}(|\ln h|^{n+1}h^{-n} + |\ln \theta|) + |\ln(\theta h^{n_1})|$$

Hence, since $n_1 > n$, we have proved the existence of some uniform constant $C \geq 1$, such that,

$$\ln \|Q(z)\| \leq \ln C + C|\ln(\theta h^{n_1})| \quad \text{ for } z \in \check{\mathcal{W}}_\theta(J) \text{ and } h \in (0, 1/C].$$

Coming back to P_θ^μ, this means that, for $z \in \check{\mathcal{W}}_\theta(J)$ different from $\rho_1, \dots, \rho_N$, we have,

$$|\Psi(z)|\, \|(P_\theta^\mu - z)^{-1}\| \leq C(\theta h^{n_1})^{-C} \prod_{\ell=1}^{N} |z - \rho_\ell|^{-1}.$$

On the other hand, if $\mathrm{dist}\,(\mathrm{Re}\, z, \mathbb{R} \setminus J) \geq 2\check{C}^{1/2}\omega_h(\theta)$, and $|\mathrm{Im}\, z| \leq 2\theta h^{n_1}$, then, writing $z = s + it$, we see that,

$$\Psi(z) = \theta e^{t^2/\theta^2} \int_{(a-s)/\theta}^{(b-s)/\theta} e^{-u^2 + 2i(t/\theta)u} du.$$

Now, $|t/\theta| \leq 2h^{n_1} \to 0$ uniformly, and we see that,

$$(a - s)/\theta \leq \widetilde{C}^{-1} - \check{C}^{1/2}\omega_h(\theta)/\theta \leq \widetilde{C}^{-1} - (h^{-n}|\ln h|)^{1/2} \to -\infty, \text{ uniformly.}$$

In the same way, we have $(b - s)/\theta \to +\infty$ uniformly as $h \to 0_+$. Therefore, we easily conclude that,

$$|\Psi(z)| \geq \frac{\theta}{C},$$

when $h \in (0, 1/C]$, with some new uniform constant $C > 0$.

As a consequence, using also that $\theta \leq h^\delta$, we finally obtain,

Proposition 6.6. *There exists a constant $C_0 \geq 1$, such that, for all $\widetilde{\mu}$, μ, I and J verifying (2.6) – (2.7), the property $\mathcal{P}(\widetilde{\mu}, \mu; I, J)$ implies,*

$$\|(P_\theta^\mu - z)^{-1}\| \leq C_0\theta^{-C_0} \prod_{\ell=1}^{N} |z - \rho_\ell|^{-1}, \tag{6.17}$$

for all $z \in J' + i[-2\theta h^{n_1}, 2\theta h^{n_1}]$, and for all $h \in (0, 1/C_0]$, where,

$$J' = \{\lambda \in \mathbb{R}\ ;\ \mathrm{dist}\,(\lambda, \mathbb{R} \setminus J) \geq C_0\omega_h(\theta)\}.$$

Since $J' = J + \mathcal{O}(\omega_h(\theta))$, Theorem 2.1 is proved.

7 Proof of Theorem 2.2

Suppose $\mathcal{P}(\widetilde{\mu}, \mu; I, J)$ holds, and $\widetilde{\mu} \geq \mu^{N_0}$ for some constant $N_0 \geq 1$. Then, for any $\theta \in [\mu^{N_0}, \widetilde{\mu}]$, any constant $N_1 \geq 1$, and any $\mu' \in [\max(\theta, \mu^{N_1}), \mu^{1/N_1}]$, we can write,

$$z - P_\theta^{\mu'} = (z - P_\theta^{\mu})(1 + (z - P_\theta^{\mu})^{-1} W), \tag{7.1}$$

with,

$$W := P_\theta^{\mu} - P_\theta^{\mu'} = V^{\mu}(x + iA_\theta(x)) - V^{\mu'}(x + iA_\theta(x)) = \mathcal{O}(\mu^\infty \langle x \rangle^{-\nu}), \tag{7.2}$$

uniformly (see Section 4). Moreover, taking J' as in Proposition 6.6, we have,

Lemma 7.1. *Let $N \geq 1$ be a constant, such that $N \geq \#\Gamma(\widetilde{\mu}, \mu, J)$ for all h small enough. Then, for any $\theta \in [\mu^{N_0}, \widetilde{\mu}]$, there exists $\tau \in [\theta h^{n_1}, 2\theta h^{n_1}]$, such that,*

$$\operatorname{dist}(\partial(J' + i[-\tau, \tau]), \Gamma(\widetilde{\mu}, \mu, J)) \geq \frac{\theta h^{n_1}}{4N}. \tag{7.3}$$

Here, $\partial(J' + i[-\tau, \tau])$ stands for the boundary of $J' + i[-\tau, \tau]$.

Proof. If it were not the case, using $\mathcal{P}(\widetilde{\mu}, \mu; I, J)$, we see that, for all $t \in [-2\theta h^{n_1}, -\theta h^{n_1}]$, there should exist $\rho \in \Gamma(\widetilde{\mu}, \mu, J)$, such that,

$$|t - \operatorname{Im} \rho| \leq \frac{\theta h^{n_1}}{4N}.$$

That is, we would have,

$$[-2\theta h^{n_1}, -\theta h^{n_1}] \subset \bigcup_{\rho \in \Gamma(\widetilde{\mu}, \mu, J)} \big[\rho - \frac{\theta h^{n_1}}{4N}, \rho + \frac{\theta h^{n_1}}{4N}\big],$$

which, again, is not possible because of the respective size of these two sets. $\square$

Remark 7.2. *With a similar proof, we see that the result of Lemma 7.1 remains valid if one replaces the interval $[\theta h^{n_1}, 2\theta h^{n_1}]$ by $[\theta h^{n_1}, \theta h^{n_1} + (\theta h^{n_1})^M]$, and one substitutes $(\theta h^{n_1})^M$ to θh^{n_1} in (7.3), where $M \geq 1$ is any arbitrary fixed number.*

Using Lemma 7.1 and Theorem 2.1, we see that, for any $z \in \partial(J' + i[-\tau, \tau])$, we have,

$$\|(P_\theta^{\mu} - z)^{-1}\| \leq C_1 \theta^{-C_1} \leq C_1 \mu^{-C_1 N_0},$$

with some new uniform constant C_1, and thus, by (7.1) and (7.2), $z - P_\theta^{\mu'}$ is invertible, too, for $z \in \partial(J' + i[-\tau, \tau])$, and the two spectral projectors,

$$\begin{aligned} \Pi &:= \frac{1}{2i\pi} \oint_{\partial(J' + i[-\tau, \tau])} (z - P_\theta^{\mu})^{-1} dz; \\ \Pi' &:= \frac{1}{2i\pi} \oint_{\partial(J' + i[-\tau, \tau])} (z - P_\theta^{\mu'})^{-1} dz, \end{aligned} \tag{7.4}$$

are well-defined and verify,

$$\|\Pi - \Pi'\| = \mathcal{O}(\mu^\infty). \tag{7.5}$$

In particular, Π and Π' have the same rank ($\leq N$), and one has,

$$\|P_\theta^\mu \Pi - P_\theta^{\mu'} \Pi'\| = \mathcal{O}(\mu^\infty). \tag{7.6}$$

Therefore, by standard finite dimensional arguments, the two sets $\sigma(P_\theta^{\mu'}) \cap (J' + i[-\tau, \tau])$ and $\sigma(P_\theta^\mu) \cap (J' + i[-\tau, \tau])$ coincide up to $\mathcal{O}(\mu^\infty)$ uniformly, and Theorem 2.2 follows.

8 Proof of Theorem 2.5

Now, for any integer $k \geq 0$, we set,

$$\mu_k := h^{kn_1}\widetilde{\mu}.$$

Since $\mathcal{P}(\widetilde{\mu}, \mu; I, J)$ holds, we can apply Theorem 2.2 with $\mu' \in [\mu_1, \mu_0]$, and deduce the existence of $J_1 \subset J$, with $J_1 = J + \mathcal{O}(\omega_h(\mu_0))$ and $I_1 \supset I$ with $I_1 = I + \mathcal{O}(\mu_0^\infty)$, independent of μ', such that, $\mathcal{P}(h^{n_1}\mu', \mu'; I_1, J_1)$ holds. In particular, $\mathcal{P}(\mu_1, \mu_0; I_1, J_1)$ holds, and we can apply Theorem 2.2 again, this time with $\mu' \in [\mu_2, \mu_1]$. Iterating the procedure, we see that, for any $k \geq 0$, there exists,

$$I_{k+1} = I_k + \mathcal{O}(\mu_k^\infty),\ J_{k+1} = J_k + \mathcal{O}(\omega_h(\mu_k))$$

hence,

$$I_{k+1} = I + \mathcal{O}(\mu_0^\infty + \cdots + \mu_k^\infty),\ J_{k+1} = J + \mathcal{O}(\omega_h(\mu_0) + \cdots + \omega_h(\mu_k)),$$

where the $\mathcal{O}$'s are also uniform with respect to k, such that $\mathcal{P}(h^{n_1}\mu', \mu'; I_{k+1}, J_{k+1})$ holds for all $\mu' \in [\mu_{k+1}, \mu_k]$.

Since the two series $\sum_k \omega_h(\mu_k) = \mathcal{O}(\omega_h(\widetilde{\mu}))$ and $\sum_k \mu_k^M = \mathcal{O}(\mu^M)$ ($M \geq 1$ arbitrary) are convergent, one can find $I' = I + \mathcal{O}(\mu^\infty)$ and $J' = J + \mathcal{O}(\omega_h(\widetilde{\mu}))$, such that,

$$I' \supset \bigcup_{k\geq 0} I_k \quad ; \quad J' \subset \bigcap_{k\geq 0} J_k.$$

Then, by construction, $\mathcal{P}(h^{n_1}\mu', \mu'; I', J')$ holds for all $\mu' \in (0, \widetilde{\mu}]$, and Theorem 2.5 is proved.

9 Proof of Theorem 2.6 – The set of resonances

From the proof of Theorem 2.5 (and with the same notations) we learn that, for all $k \geq 0$, $\mathcal{P}(\mu_{k+1}, \mu_k; I_{k+1}, J_{k+1})$ holds. Therefore, applying Theorem 2.2 with $\theta = \mu' = \mu_{k+1}$, we obtain that there exist $\tau_{k+2} \in [\mu_{k+2}, 2\mu_{k+2}]$, $J'_{k+1} = J_{k+1} + \mathcal{O}(\omega_h(\mu_{k+1}))$, and a bijection,

$$b_k : \Gamma(P^{\mu_k}) \cap (J'_{k+1} - i[0, \tau_{k+2}]) \to \Gamma(P^{\mu_{k+1}}) \cap (J'_{k+1} - i[0, \tau_{k+2}])$$

such that,

$$b_k(\lambda) - \lambda = \mathcal{O}(\mu_k^\infty) \text{ uniformly.} \tag{9.1}$$

In addition, we deduce from the proof of Theorem 2.2 (in particular Lemma 7.1), that the τ_k's can be chosen in such a way, that,

$$\operatorname{dist}(\partial(J'_{k+1} + i[-\tau_{k+2}, \tau_{k+2}]), \Gamma(P^{\mu_k})) \geq \frac{\mu_k^C}{C}, \tag{9.2}$$

for some constant $C > 0$. Setting

$$\Lambda_k := \Gamma(P^{\mu_k}) \cap (J'_{k+1} - i[0, \tau_{k+2}]),$$

where the elements are repeated according to their multiplicity, and, starting from an arbitrary element λ_j of Λ_0 ($1 \leq j \leq N := \#\Lambda_0 = \mathcal{O}(1)$), we distinguish two cases.

-Case A: For all $k \geq 0$, $b_k \circ b_{k-1} \circ \cdots \circ b_0(\lambda_j) \in \Lambda_{k+1}$.

In that case, we can consider the sequence defined by,

$$\lambda_{j,k} := b_k \circ b_{k-1} \circ \cdots \circ b_0(\lambda_j),$$

($k \geq 0$), and, using (9.1), we see that, for any $k > \ell \geq 0$, we have,

$$|\lambda_{j,k} - \lambda_{j,\ell}| \leq \sum_{m=\ell}^{k-1} |\lambda_{j,m+1} - \lambda_{j,m}| \leq C_1 \sum_{m=\ell}^{k-1} \mu_{m+1} \leq C_1 \mu_0 \frac{h^{n_1 \ell}}{1 - h^{n_1}},$$

so that $(\lambda_{j,k})_{k \geq 1}$ is a Cauchy sequence, and we set,

$$\rho_j := \lim_{k \to +\infty} \lambda_{j,k}.$$

Notice that according to this definition, we have a natural notion of multiplicity of a resonance ρ, namely the number of sequences ρ_j converging to ρ.

-Case B: There exists $k_j \geq 0$ such that $b_{k-1} \circ \cdots \circ b_0(\lambda_j) \in \Lambda_k$ for all $k \leq k_j$, while $b_{k_j} \circ \cdots \circ b_0(\lambda_j) \notin \Lambda_{k_j+1}$. (Here, and in the sequel, we use the convention of notation: $b_{-1} \circ b_0 := Id$.)

Then, we set,

$$\rho_j := b_{k_j} \circ \cdots \circ b_0(\lambda_j).$$

In particular, since, by construction, $\operatorname{Re} \rho_j \in I_{k_j+2} \subset J_{k_j+1}$, and $\rho_j \notin \Lambda_{k_j+1}$, we see that, necessarily, $\operatorname{Im} \rho_j \in [-\tau_{k_j+2}, -\tau_{k_j+3})$.

Moreover, if, in Case A, we set $k_j := +\infty$, then, for any $j = 1, \ldots, \#\Lambda_0$ and $k \geq 0$, in both cases we have the equivalence,

$$|\operatorname{Im} \rho_j| \leq \tau_{k+2} \iff k \leq k_j. \tag{9.3}$$

Now, if $\mu' \in (0, \widetilde{\mu}]$, then $\mu' \in (\mu_{k+1}, \mu_k]$ for some $k \geq 0$, and Theorem 2.2 tells us that $\Gamma(P^{\mu'}) \cap (J'_{k+1} - i[0, \tau_{k+2}])$ coincides with Λ_k up to $\mathcal{O}(\mu_k^\infty)$ $(= \mathcal{O}((\mu')^\infty))$. Therefore, setting,

$$\Lambda := \{\rho_1, \ldots, \rho_N\},$$

the first part of Theorem 2.6 will be proved if we can show the existence, for any $k \geq 0$, of a bijection,

$$\widetilde{b}_k \,:\, \Lambda \cap (J'_{k+1} - i[0, \tau_{k+2}]) \to \Lambda_k,$$

such that $\widetilde{b}_k(\rho) - \rho = \mathcal{O}(\mu_k^\infty)$ uniformly. (Actually, we do not necessarily have $\tau_{k+2} \in [h^{2n_1}\mu', 2h^{2n_1}\mu']$, but rather, $\tau_{k+2} \in [h^{2n_1}\mu', 2h^{n_1}\mu')$. However, if $\tau_{k+2} \geq 2h^{2n_1}\mu'$, an argument similar to that of Lemma 6.3 or Lemma 7.1 gives the result stated in Theorem 2.6.)

By construction, we have,

$$\Lambda_k = \{b_{k-1} \circ \cdots \circ b_0(\lambda_j)\,;\, j = 1, \cdots, N \text{ such that } k_j \geq k\}.$$

while, by (9.3),

$$\Lambda \cap (J'_{k+1} - i[0, \tau_{k+2}]) = \{\rho_j\,;\, j = 1, \cdots, N \text{ such that } k_j \geq k\}.$$

Then, for all j verifying $k_j \geq k$, we set,

$$\widetilde{b}_k(\rho_j) := b_{k-1} \circ \cdots \circ b_0(\lambda_j),$$

so that $\widetilde{b}_k$ defines a bijection between $\Lambda \cap (J'_{k+1} - i[0, \tau_{k+2}])$ and Λ_k. Moreover, in Case A, for any $M \geq 1$, we have,

$$|\widetilde{b}_k(\rho_j) - \rho_j| = \lim_{\ell \to +\infty} |b_\ell \circ \cdots \circ b_k(\widetilde{b}_k(\lambda_j)) - \widetilde{b}_k(\lambda_j)| \leq \sum_{m=k}^{+\infty} C_M \mu_m^M = \frac{C_M \mu_k^M}{1 - h^{n_1}},$$

while, in Case B, we obtain,

$$|\widetilde{b}_k(\rho_j) - \rho_j| = |b_{k_j} \circ \cdots \circ b_k(\widetilde{b}_k(\lambda_j)) - \widetilde{b}_k(\lambda_j)| \leq \sum_{k \leq m \leq k_j} C_M \mu_m^M \leq \frac{C_M \mu_k^M}{1 - h^{n_1}},$$

(with the usual convention $\sum_{m \in \emptyset} := 0$). Therefore, in both cases, for $h > 0$ small enough, we find,

$$|\widetilde{b}_k(\rho_j) - \rho_j| \leq 2C_M \mu_k^M,$$

and this gives the first part of Theorem 2.6.

Concerning the second part of Theorem 2.6, let $\widetilde{\Lambda}$ be another set verifying $(\star)$. In particular, for any $k \geq 0$, there exist $\tau_{k+2}, \widetilde{\tau}_{k+2} \in [\mu_{k+2}, 2\mu_{k+2}]$, such that, $\widetilde{\Lambda} \cap (J'_{k+1} - i[0, \widetilde{\tau}_{k+2}])$ (resp. $\Lambda \cap (J'_{k+1} - i[0, \tau_{k+2}])$) coincides with, $\widetilde{\Lambda}_k := \Gamma(P^{\mu_k}) \cap (J'_{k+1} - i[0, \widetilde{\tau}_{k+2}])$, (resp. Λ_k), up to $\mathcal{O}(\mu_k^\infty)$.

Therefore, taking $k = 0$, and using again an argument similar to that of Lemma 6.3 or Lemma 7.1, that gives the existence of $\tau' \in [\frac{1}{2}\mu_2, \mu_2]$ and $C > 0$ constant, such that,

$$\operatorname{dist}(\partial(J'_1 + i[-\tau', \tau']), \Gamma(P^{\mu_0})) \geq \frac{\mu_0^C}{C}, \tag{9.4}$$

we obtain that the two sets $\Lambda \cap (J'_1 - i[0, \tau'])$ and $\widetilde{\Lambda} \cap (J'_1 - i[0, \tau'])$ coincide up to $\mathcal{O}(\mu_0^\infty)$, and thus have same cardinal. For $k \geq 0$, we denote by,

$$\begin{aligned} B_k \,&:\, \Lambda_k \,\to\, \Lambda \cap (J'_{k+1} - i[0, \tau_{k+2}]); \\ \widetilde{B}_k \,&:\, \widetilde{\Lambda}_k \,\to\, \widetilde{\Lambda} \cap (J'_{k+1} - i[0, \widetilde{\tau}_{k+2}])\}, \end{aligned}$$

the corresponding bijections. Then, thanks to (9.4), we can consider the bijection,

$$\varphi_0 = \widetilde{B}_0 \circ B_0^{-1}\big|_{\Lambda\cap(J_1'-i[0,\tau'])} \,:\, \Lambda \cap (J_1' - i[0,\tau']) \to \widetilde{\Lambda} \cap (J_1' - i[0,\tau']) :$$

Using (9.2) and the fact that $\widetilde{B}_k$ differ from the identity by $\mathcal{O}(\mu_k^\infty)$, we see that, for $k \geq 1$,

$$\operatorname{dist}(\partial(J_{k+1}' + i[-\tau_{k+2}, \tau_{k+2}]), \widetilde{\Lambda}) \geq \frac{\mu_k^C}{C}, \tag{9.5}$$

for some other constant $C > 0$.

Then setting

$$\mathcal{E}_0 := \Lambda \cap \{-\tau' \leq \operatorname{Im} z < -\tau_3\},$$

and, for $k \geq 1$,

$$\mathcal{E}_k := \Lambda \cap \{-\tau_{k+2} \leq \operatorname{Im} z < -\tau_{k+3}\}.$$

we see that, for all $k \geq 1$, the application,

$$\widetilde{B}_k \circ B_k^{-1}\big|_{\mathcal{E}_k} : \mathcal{E}_k \to \widetilde{\Lambda} \cap \{-\tau_{k+2} \leq \operatorname{Im} z < -\tau_{k+3}\}, \tag{9.6}$$

is a bijection.

Finally, for $\rho \in \Lambda \cap (J_1' - i[0,\tau'])$, we define,

- $B(\rho) = \widetilde{B}_k \circ B_k^{-1}(\rho)$, if $\rho \in \mathcal{E}_k$ for some $k \geq 0$;
- $B(\rho) = \rho$, if $\rho \in \mathbb{R}$.

We first show,

Lemma 9.1. *$\Lambda \cap \mathbb{R} = \widetilde{\Lambda} \cap \mathbb{R}$.*

Proof. We only show that any ρ in $\Lambda \cap \mathbb{R}$ is also in $\widetilde{\Lambda}$, the proof of the other inclusion being similar. For such a ρ, $B_k^{-1}(\rho) \in \Lambda_k$ is well defined for all $k \geq 1$, and since B_k^{-1} differs from the identity by $\mathcal{O}(\mu_k^\infty)$, we obtain,

$$\alpha_k := B_k^{-1}(\rho) \to \rho \quad \text{as} \quad k \to +\infty.$$

On the other hand, since $\Lambda_{k+1} \subset \widetilde{\Lambda}_k = \widetilde{B}_k^{-1}(\widetilde{\Lambda})$, there exists some $\widetilde{\rho}_k \in \widetilde{\Lambda}$ such that $\alpha_{k+1} = \widetilde{B}_k^{-1}(\widetilde{\rho}_k)$. By taking a subsequence, we can assume that $\widetilde{\rho}_k$ admits a limit $\widetilde{\rho} \in \widetilde{\Lambda}$ as $k \to +\infty$. Then, using that $\widetilde{B}_k^{-1}$ differs from the identity by $\mathcal{O}(\mu_k^\infty)$, we also obtain,

$$\alpha_{k+1} \to \widetilde{\rho} \quad \text{as} \quad k \to +\infty.$$

Therefore, we deduce that $\rho = \widetilde{\rho} \in \widetilde{\Lambda}$ and the lemma is proved. □

Using Lemma 9.1, we see that the application B is well defined from $\Lambda \cap (J_1' - i[0, \tau'])$ to $\widetilde{\Lambda} \cap (J_1' - i[0, \tau'])$. Moreover, if $\rho \in \mathcal{E}_k$ for some $k \geq 0$, we have,

$$|B(\rho) - \rho| = |\widetilde{B}_k \circ B_k^{-1}(\rho) - \rho| = \mathcal{O}(\mu_k^\infty),$$

and, since $\tau_{k+3} \leq |\mathrm{Im}\ \rho| \leq \tau_{k+2} = \mathcal{O}(h^{2n_1})$, we also have,

$$\mu_k \leq h^{-3n_1} \tau_{k+3} \leq h^{-3n_1} |\mathrm{Im}\ \rho| \leq C |\mathrm{Im}\ \rho|^{1/C},$$

where $C > 0$ is a large enough constant. Thus, we always have,

$$|B(\rho) - \rho| = \mathcal{O}(|\mathrm{Im}\ \rho|^\infty).$$

Therefore, it just remains to see that B is a bijection, but this is an obvious consequence of (9.6), Lemma 9.1, and the definition of B. Thus Theorem 2.6 is proved.

10 Shape resonances

Here we prove Theorem 3.1. Under the assumptions of Section 3, one can construct, as in [8], a function $G_1 \in C^\infty(\mathbb{R}^{2n})$, supported near $p^{-1}([\lambda_0 - 2\varepsilon, \lambda_0 + 2\varepsilon]) \backslash \{x_0\}$ for some $\varepsilon > 0$, such that,

$$G_1(x, \xi) = x \cdot \xi \text{ for } x \text{ large enough, } |p(x, \xi) - \lambda_0| \leq \varepsilon; \tag{10.1}$$

$$H_p G_1(x, \xi) \geq \varepsilon \text{ for } x \in \mathbb{R}^n \backslash \ddot{\mathrm{O}} \text{ and } |p(x, \xi) - \lambda_0| \leq \varepsilon. \tag{10.2}$$

We also set,

$$\widetilde{P} := P + W,$$

where $W = W(x) \in C^\infty(\mathbb{R}^n)$ is a non negative function, supported in a small enough neighborhood of x_0, and such that $W(x_0) > 0$. In particular, denoting by $\widetilde{p}(x, \xi) = \xi^2 + V(x) + W(x)$ the principal symbol of $\widetilde{P}$, we have $\widetilde{p}^{-1}(\lambda_0) \subset (\mathbb{R}^n \backslash \ddot{\mathrm{O}}) \times \mathbb{R}^n$, and thus λ_0 is a non-trapping energy for $\widetilde{P}$.

Now, we take μ and $\widetilde{\mu}$ such that,

$$\mu \leq h^\delta \quad ; \quad \widetilde{\mu} \leq \min(\mu, h^{2+\delta})$$

with $\delta > 0$ arbitrary (so that $\mu, \widetilde{\mu}$ verify (2.6)), and we denote by V^μ a $|x|$-analytic $(\mu, \widetilde{\nu})$-approximation of V as before. We also set,

$$P^\mu = -h^2 \Delta + V^\mu \quad ; \quad \widetilde{P}^\mu = P^\mu + W,$$

and, if in (2.5) we take A supported away from $\mathrm{Supp}\, W$, we see that the distorted operators P_θ^μ and $\widetilde{P}_\theta^\mu$ are well defined for $0 < \theta \leq \widetilde{\mu}$. Then, we set,

$$G(x, \xi) := G_1(x, \xi) - A(x) \cdot \xi,$$

that, by (10.1), is in $C_0^\infty(\mathbb{R}^n; \mathbb{R})$, and we consider its semiclassical Weyl-quantization $G^W = \mathrm{Op}_h^W(G)$ (see 6.1).

Since $\theta/h^2 \le \widetilde{\mu}/h^2 \le h^\delta$, a straightforward computation shows that the operator,

$$R_\theta^\mu := \frac{1}{\theta}\mathrm{Im}\,\left(e^{\theta G^W/h}\widetilde{P}_\theta^\mu e^{-\theta G^W/h}\right)$$

is a semiclassical pseudodifferential operator, with symbol r_θ^μ verifying,

$$\partial^\alpha r_\theta^\mu = \mathcal{O}(\langle\xi\rangle^2) \text{ for all } \alpha \in \mathbb{N}^{2n};$$
$$r_\theta^\mu(x,\xi) = -H_{\widetilde{p}^{\widetilde{\mu}}}(A(x)\cdot\xi + G) + \mathcal{O}(h^\delta) = -H_p G_1(x,\xi) + \mathcal{O}(h^\delta),$$

uniformly with respect to $\theta \in (0,\widetilde{\mu}]$ and $h > 0$ small enough. As a consequence, using (10.2), we see that R_θ^μ is elliptic in a neighborhood of $\{p(x,\xi) + W(x) = \lambda_0\}$ (uniformly with respect to θ and μ). Then, by arguments similar to those of Section 6.1, we deduce that the operator

$$Q_\theta^\mu := e^{\theta G^W/h}\widetilde{P}_\theta^\mu e^{-\theta G^W/h}$$

verifies

$$\|(Q_\theta^\mu - z)^{-1}\| = \mathcal{O}(\theta^{-1}),$$

uniformly for $|\mathrm{Re}\, z - \lambda_0| + \theta^{-1}|\mathrm{Im}\, z|$ small enough, $\theta \in (0,\widetilde{\mu}]$, and $h > 0$ small enough. Since $\|\theta G^W/h\| \to 0$ uniformly as $h \to 0$, this also gives,

$$\|(\widetilde{P}_\theta^\mu - z)^{-1}\| = \mathcal{O}(\theta^{-1}),$$

and from this point, one can follow all the procedure used in [10] Sections 9 and 10. In particular, using the same notations as in [10], by Agmon-type inequalities we see that the distribution kernel $K_{(\widetilde{P}_\theta^\mu - z)^{-1}}$ of $(\widetilde{P}_\theta^\mu - z)^{-1}$ verifies,

$$K_{(\widetilde{P}_\theta^\mu - z)^{-1}}(x,y) = \widetilde{\mathcal{O}}(\theta^{-1}e^{-d(x,y)/h})$$

where $d(x,y)$ stands for the Agmon distance between x and y (see [10, Lemma 9.4]). Then, assuming $\theta = \tilde{\mu} \ge e^{-\eta/h}$ for some $\eta > 0$ constant small enough, and performing a suitable Grushin problem as in [10], we deduce that the resonances of P^μ in $[\lambda_0, \lambda_0 + Ch] - i[0, \lambda_0 \min(\mu, h^{2+\delta})]$ ($C > 0$ constant arbitrary) are close to the eigenvalues of the Dirichlet realization of P on $\{d(x, \mathbb{R}^n\backslash\ddot{\mathrm{O}}) \ge \eta/3)\}$, up to $\mathcal{O}(e^{-2(S_0-\eta)/h})$. Since these eigenvalues are real and admit semiclassical asymptotic expansions of the form,

$$\lambda_k \sim \lambda_0 + e_k h + \sum_{\ell\ge 1}\lambda_{k,\ell}h^{1+\frac{\ell}{2}}$$

(where the e_k's are as in Theorem 3.1), we obtain for the corresponding resonances ρ_k of P^μ,

$$\mathrm{Re}\,\rho_k \sim \lambda_0 + e_k h + \sum_{\ell\ge 1}\lambda_{k,\ell}h^{1+\frac{\ell}{2}} \quad ; \quad \mathrm{Im}\,\rho_k = \mathcal{O}(e^{-2(S_0-\eta)/h}), \tag{10.3}$$

uniformly. In particular, taking μ and $\widetilde{\mu}$ as in Theorem 3.1, the result easily follows. Moreover, since the previous discussion can be applied to any $\mu' \in [e^{-\eta/h}, h^\delta]$, application of Theorem 2.6 tells us that the resonances of P in $[\lambda_0, \lambda_0 + Ch] - i[0, \frac{1}{2}h^{2n+\max(\frac{n}{2},1)+1+3\delta}]$ satisfy to the same estimates (10.3).

A Appendix

A.1 Proof of Lemma 5.1

We denote by χ_0 a real smooth function on $\mathbb{R}$ verifying,

- $\chi_0(s) = 0$ for $s \leq 0$;
- $\chi_0(s) = 1$ for $s \geq \ln 2$;
- χ_0 is non decreasing.

Then, for $r \geq 0$, we set,

$$G(r) := \chi_0(r - R_0)(1 - \chi_0(r - \ln\lambda))e^r + 2\lambda\chi_0(r - \ln\lambda),$$

and,

$$g(r) := \int_0^r G(s)ds.$$

In particular, g verifies Condition (i) of Lemma 5.1, and we have,

- $G(r) = \chi_0(r - R_0)e^r$ for $r \in [R_0, \ln\lambda]$;
- $G(r) = (1 - \chi_0(r - \ln\lambda))e^r + 2\lambda\chi_0(r - \ln\lambda)$ for $r \in [\ln\lambda, \ln 2\lambda]$;
- $G(r) = 2\lambda$ for $r \in [\ln 2\lambda, +\infty)$.

Thus, $g' = G \leq 2\lambda$ and $g''(r) = G'(r) \geq 0$ on $\mathbb{R}_+$ (this is immediate on $[R_0, \ln\lambda] \cup [\ln 2\lambda, +\infty)$, while, on $[\ln\lambda, \ln 2\lambda]$, we compute, $G'(r) = (1 - \chi_0(r - \ln\lambda))e^r + \chi_0'(r - \ln\lambda)(2\lambda - e^r) \geq 0$).

Therefore, g is convex on $\mathbb{R}_+$, so that Condition (iii) of Lemma 5.1 is verified by g, too, while Condition (v) is obvious.

As for condition (iv), we observe,

- On $[0, R_0 + \ln 2]$, one has, $g' + |g''| = \mathcal{O}(1)$;
- On $[R_0 + \ln 2, \ln\lambda]$, one has, $g(r) \geq \int_{R_0+\ln 2}^r e^s ds = e^r - 2e^{R_0}$, while $g'(r) = g''(r) = e^r \leq g(r) + 2e^{R_0}$;
- On $[\ln\lambda, +\infty)$, one has, $g(r) \geq g(\ln\lambda) = \lambda$, and thus $g' + |g''| = \mathcal{O}(g)$.

So, g verifies Conditions (ii)-(v) of Lemma 5.1.

For $r \in [\ln 2\lambda, +\infty)$, we have,

$$g(r) = g(\ln 2\lambda) + 2\lambda(r - \ln 2\lambda) = 2\lambda r - \alpha_\lambda, \tag{A.1}$$

where $\alpha_\lambda := 2\lambda\ln 2\lambda - g(\ln 2\lambda)$, and, since,

$$g(\ln 2\lambda) \leq \int_0^{\ln\lambda} e^r dr + \int_{\ln\lambda}^{\ln 2\lambda} 2\lambda dr = (1 + 2\ln 2)\lambda;$$

$$g(\ln 2\lambda) \geq \int_{R_0+\ln 2}^{\ln 2\lambda} e^r dr \geq 2\lambda - 2e^{R_0}.$$

we see that,

$$2\lambda \ln 2\lambda - (1 + 2\ln 2)\lambda \leq \alpha_\lambda \leq 2\lambda \ln 2\lambda - 2\lambda + 2e^{R_0}.$$

Therefore, for λ large enough, the unique point r_λ, solution of $g(r_\lambda) = \lambda r_\lambda$, is given by,

$$r_\lambda = \frac{\alpha_\lambda}{\lambda} \in [2\ln\lambda - 1, 2\ln\lambda - 2 + 2\ln 2 + 2\lambda^{-1}e^{R_0}] \subset [2\ln\lambda - 1, 2\ln\lambda - \varepsilon_0], \tag{A.2}$$

where $\varepsilon_0 := 1 - \ln 2 > 0$.

Now, we fix some real-valued function $\varphi_0 \in C^\infty(\mathbb{R})$, such that,

- $\varphi_0(s) = 2s$ for $s \leq -\varepsilon_0$;
- $\varphi_0(s) = s$ for $s \geq \varepsilon_0$;
- $1 \leq \varphi_0' \leq 2$ everywhere.

Then, using (A.1)-(A.2), we see that the function f_λ defined by,

- $f_\lambda(r) := g(r)$ for $r \in [0, \ln 2\lambda]$;
- $f_\lambda(r) := \lambda\varphi_0(r - r_\lambda) + \alpha_\lambda$ for $r \geq \ln 2\lambda$,

is smooth on $\mathbb{R}_+$, and verifies all the conditions required in Lemma 5.1. □

A.2 The distorted Laplacian

Lemma A.1. *If $\theta > 0$ is small enough, the function Φ_θ defined in (5.2) verifies,*

$$\text{Im}\,[({}^t d\Phi_\theta(x))^{-1}\xi]^2 \leq -\theta a(|x|)|\xi|^2.$$

for all $(x, \xi) \in \mathbb{R}^{2n}$.

Proof. Let $F := {}^t dA = dA = (F_{i,j})_{1\leq i,j\leq n}$. We compute,

$$F_{i,j}(x) = a(x)\delta_{i,j} + a'(|x|)\frac{x_i x_j}{|x|},$$

that is, denoting by $\pi_x := |x|^{-2}x \cdot {}^t x$ the orthogonal projection onto $\mathbb{R}x$, and recalling the notation $b(r) = ra(r)$,

$$F(x) = a(|x|)I + a'(|x|)|x|\pi_x = b'(|x|)\pi_x + a(|x|)(I - \pi_x).$$

In particular, using Lemma 5.1, we obtain,

$$0 \leq a(|x|) \leq F(x) \leq 2,$$

in the sense of self-adjoint matrices. On the other hand, we have,

$$({}^t d\Phi_\theta(x))^2 = (I + i\theta F(x))^2 = S_\theta + iT_\theta,$$

with $S_\theta = I - \theta^2 F(x)^2$ and $T_\theta = 2\theta F(x)$. Hence, $T_\theta \geq 0$, and, since S_θ, T_θ and F commute, an easy computation gives,

$$\mathrm{Im}\,[({}^t d\Phi_\theta(x))^{-1}\xi]^2 \quad = \quad -T_\theta(S_\theta^2 + T_\theta^2)^{-1}\xi\cdot\xi = -2\theta F(1+\theta^2F^2)^{-2}\xi\cdot\xi.$$

As a consequence, for θ small enough, we find,

$$\mathrm{Im}\,[({}^t d\Phi_\theta(x))^{-1}\xi]^2 \leq -\theta F(x)\xi\cdot\xi \leq -\theta a(|x|)|\xi|^2.$$

$\square$

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